

RESOLUTION OF NON-SINGULARITIES FOR MUMFORD CURVES

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INTRODUCTION

In this article, given a hyperbolic curve X over $\overline{\mathbf{Q}}_p$, we are interested in finding a finite étale cover Y of this curve such that the stable reduction of the cover has irreducible components lying over the smooth locus of the stable reduction of X . Such techniques of resolution of nonsingularities are often used in anabelian geometry. We will apply our results to the anabelian study of the tempered fundamental group.

In [13, th. 0.2], A. Tamagawa proved that for every hyperbolic curve $X = \overline{X} \setminus D$ and every closed point x of the stable reduction of X , there exists a finite étale cover Y and an irreducible component y of the stable reduction of Y lying above x . We would like to generalize this to all the semistable reductions of X : given a semistable model \mathcal{X} of X and a closed point x of the special fiber \mathcal{X}_s of \mathcal{X} , is there a finite étale cover Y and an irreducible component y of the minimal semistable model \mathcal{Y} of Y above \mathcal{X} such that y lies above x ? To give an example of anabelian motivation for this kind of resolution of nonsingularities, as shown by F. Pop and J. Stix in [12, cor. 41], if X_0 is a geometrically connected hyperbolic curve over a finite extension K of \mathbf{Q}_p such that $X_{0, \overline{\mathbf{Q}}_p}$ satisfies this kind of resolution of nonsingularities, every section of $\pi_1^{\text{alg}}(X_0) \rightarrow \text{Gal}(\overline{\mathbf{Q}}_p/K)$ has its image in a decomposition group of a unique valutive point. In this article, we will prove that this resolution of nonsingularities is satisfied by Mumford curves.

First, let us translate this in an analytic setting. Let X^{an} be the Berkovich space of X . Given a semistable model \mathcal{X} of X , there is a reduction map $\pi_{\mathcal{X}} : X^{\text{an}} \rightarrow \mathcal{X}_s$. If η_y is the generic point of an irreducible component of \mathcal{Y} , the subset $\pi_{\mathcal{Y}}^{-1}(\eta_y) \subset Y^{\text{an}}$ is reduced to a single point. We denote by $V(\mathcal{Y})$ the set of points of Y^{an} whose image by $\pi_{\mathcal{Y}}$ is a generic point. Therefore our question reduces to the following: is there Y and an element of $V(\mathcal{Y})$ which is mapped to $\pi_{\mathcal{X}}^{-1}(x)$? Since $V(\mathcal{Y})$ contains $V(Y) := V(\mathcal{Y}_0)$ where \mathcal{Y}_0 is the stable model of Y and $\pi_{\mathcal{X}}^{-1}(x)$ is a non-empty open subset of X^{an} , it is enough to show that the union $\tilde{V}(X)$ of the images of $V(Y)$ in X^{an} , where Y runs over finite étale covers of X , is dense in X^{an} .

Theorem 0.1 (th. 2.6). *Let X be a hyperbolic Mumford curve. Then $\tilde{V}(X)$ is dense in X^{an} .*

To do so, we will study μ_{p^n} -torsors of X . Projective systems of μ_{p^n} -torsors of X are classified by $H^1(X, \mathbf{Z}_p(1))$. Let c be an element of $H^1(X, \mathbf{Z}_p(1))$. Let x be a \mathbf{C}_p -point of X . There is a small rigid neighborhood D of x in $X_{\mathbf{C}_p}$ isomorphic to a disk and a morphism $f : D \rightarrow \mathbf{G}_m$, i.e. an element $f \in O^*(D)$, such that the restriction of c to D is the pullback of the canonical element of $H^1(\mathbf{G}_m, \mathbf{Z}_p(1))$. Let $Y_n \rightarrow X$ be the μ_{p^n} -torsor induced by c . For n big enough, there is a smallest closed disk D_n of D centered at x such that $Y_n|_{D_n} \rightarrow X_n|_{D_n}$ is a non-trivial cover. Then the behavior of the restriction of Y_n to the Berkovich generic point x_n of D_n for n big enough only depends on the ramification index e of $f : D \rightarrow \mathbf{G}_m$ at x . More

precisely, if y_n is a preimage of x_n in Y_n , the extension $\mathcal{H}(y_n)/\mathcal{H}(x_n)$ of complete residue fields induce an extension $k(y_n)/k(x_n)$ of their reduction in characteristic p . This extension is an Artin-Schreier extension: $k(x_n)$ is isomorphic to $\overline{\mathbf{F}}_p(X)$ and $k(y_n)$ is isomorphic to $k(x_n)[Y]/(Y^p - Y - X^e)$. The general study of Artin-Schreier extensions tells us that if e is not a power of p , $k(y_n)$ is not a rational extension of $\overline{\mathbf{F}}_p$. This implies that $y_n \subset V(Y_n)$ and that $x_n \in \tilde{V}(X)$ and therefore x lies in the closure of $\tilde{V}(X)$.

By Hodge-Tate theory, one has a canonical decomposition $H^1(X, \mathbf{Z}_p(1)) \otimes_{\mathbf{Z}_p} \mathbf{C}_p = H^1(X_{\mathbf{C}_p}, \mathcal{O}_{X_{\mathbf{C}_p}})(1) \oplus H^0(X_{\mathbf{C}_p}, \Omega_{X_{\mathbf{C}_p}})$. Let us consider the induced map $p : H^1(X, \mathbf{Z}_p(1)) \rightarrow H^0(X_{\mathbf{C}_p}, \Omega_{X_{\mathbf{C}_p}})$. Assume now X is a Mumford curve over $\overline{\mathbf{Q}}_p$. Then the image of p lies in $H^0(X, \Omega_X)$ and, for $c \in H^1(X, \mathbf{Z}_p(1))$, the restriction of $p(c)$ to D is $\frac{df}{f}$. Let $\Omega \subset \mathbf{P}^1$ be the universal topological cover of X . If $x \in \Omega(\overline{\mathbf{Q}}_p)$, one can find a rational function f with no poles nor zero in Ω such that $\frac{df}{f}$ has a zero at x with multiplicity m such that $m + 1$ is not a power of p . Let c_f be the pullback of the canonical element of $H^1(\mathbf{G}_m, \mathbf{Z}_p(1))$ along $f : \Omega \rightarrow \mathbf{G}_m$, and let $x_n \in \Omega$ be defined as previously. Then for the topology of the uniform convergence on every compact on $O^*(D)$, we will approximate f by elements of $\bigcup_{X'} \Theta(X')$, where X' runs over finite topological pointed covers of X and $\Theta(X')$ is the set of theta functions of X' . Using this, one can construct for every n a finite topological cover X' of X and a μ_{p^n} -torsor $Y \rightarrow X'$ such that the preimage in Y of the image in X' of x_n lies in $V(Y)$. Therefore $x \in \tilde{V}(X)$. Since $X_{\overline{\mathbf{Q}}_p}$ is dense in X^{an} , one gets the density of $\tilde{V}(X)$.

In a second part, we use the resolution of nonsingularities to study the tempered fundamental group. One shows the following:

Theorem 0.2 (th. 3.9). *Let X_1 and X_2 be two Mumford curves over $\overline{\mathbf{Q}}_p$. Given an isomorphism between their tempered fundamental groups, there is a canonical homeomorphism between there Berkovich spaces.*

The strategy is the following. For every semistable model of a curve X , there is a retraction from \overline{X}^{an} to the graph of this semistable reduction. One gets a map from \overline{X}^{an} to the projective limit of graphs of semistable reductions, which is a homeomorphism. If X is a Mumford curve, by resolution of nonsingularities, semistable models of the form \mathcal{Y}/G , where Y runs over finite Galois cover of X , \mathcal{Y} is the stable model of Y and $G = \text{Gal}(Y/X)$, are cofinal among semistable models of X . However, a theorem of S. Mochizuki tells us that one can recover the graph of the stable reduction from the tempered fundamental group ([11, cor. 3.11]). Therefore if Y_1 is a finite Galois cover of X_1 and Y_2 is the corresponding finite Galois cover given by the isomorphism of tempered fundamental groups, the graph \mathbb{G}_{Y_1} of \mathcal{Y}_1 is canonically isomorphic to the graph \mathbb{G}_{Y_2} of \mathcal{Y}_2 , and one gets a similar isomorphism after quotienting by $\text{Gal}(Y_1/X_1) \simeq \text{Gal}(Y_2/X_2)$. The problem is to recover from the tempered fundamental group the transition maps between the geometric realisation of the different graphs.

At the end of the article, we will be interested by the anabelianness of the tempered fundamental group for punctured Tate curves:

Theorem 0.3 (th. 4.1). *Let $q_1, q_2 \in \overline{\mathbf{Q}}_p$ such that $|q_1|, |q_2| < 1$. Assume there exists an isomorphism ψ between the tempered fundamental groups of $(\mathbf{G}_m/q_1^{\mathbf{Z}})\backslash\{1\}$ and $(\mathbf{G}_m/q_2^{\mathbf{Z}})\backslash\{1\}$. Then there exists $\sigma \in \text{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ such that $q_2 = \sigma(q_1)$.*

However, the proof of this result does not build any particular element of $\text{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$.

Let $\Omega_i = \mathbf{G}_m \setminus \{q_i^n\}_{n \in \mathbf{Z}}$ and $X_i = \Omega_i/q_i^{\mathbf{Z}}$. According to theorem 0.2, the isomorphism of tempered fundamental groups induces a homeomorphism $\bar{\psi} : \mathbf{G}_m^{\text{an}} \rightarrow \mathbf{G}_m^{\text{an}}$ which maps q_1^n to q_2^n for every $n \in \mathbf{Z}$.

Elements of $O(\Omega_i)^*$ correspond, up to a scalar, to a current on the semitree \mathbb{T}_i of Ω_i . Since ψ induces an isomorphism $\mathbb{T}_1 \simeq \mathbb{T}_2$, one gets a group isomorphism $\alpha : O(\Omega_1)^* \rightarrow O(\Omega_2)^*$. The crucial point will be to show that for every $f \in O^*(\Omega_1)$ and $z \in \mathbf{G}_m(\mathbf{C})$, the multiplicity of $\frac{df}{f}$ at z coincide with the multiplicity of $\frac{d\alpha(f)}{\alpha(f)}$ at $\bar{\psi}(z)$. By density of \mathbf{Z} in \mathbf{Z}_p , one also gets a similar result for \mathbf{Z}_p -linear combinations of differential 1-forms as $\frac{df}{f}$. Once one knows this, one can build, for every $n \in \mathbf{N}$, an element $f \in O^*(\Omega_1)$ such that $\frac{df}{f}(1) = q_1^n$ and $\frac{d\alpha(f)}{\alpha(f)}(1) = q_2^n$: one therefore gets that for every polynomial $P \in \mathbf{Z}_p[X]$, $P(q_1) = 0$ if and only if $P(q_2) = 0$.

1. BERKOVICH GEOMETRY OF CURVES

In the following $K = \mathbf{C}_p$, k is its residue field, which is isomorphic to $\bar{\mathbf{F}}_p$. The norm will be chosen so that $|p| = p^{-1}$ and the valuation so that $v(p) = 1$. All valued fields will have valuations with values in $\mathbf{R}_{\geq 0}$.

If X is an algebraic variety over K , one can associate to X a topological set X^{an} with a continuous map $\phi : X^{\text{an}} \rightarrow X$ defined in the following way. A point of X^{an} is an equivalence class of morphisms $\text{Spec } K' \rightarrow X$ over $\text{Spec } K$ where K' is a complete valued extension of K . Two morphisms $\text{Spec } K' \rightarrow X$ and $\text{Spec } K'' \rightarrow X$ are equivalent if there exists a common valued extension L of K' and K'' such that

$$\begin{array}{ccc} \text{Spec } L & \longrightarrow & \text{Spec } K'' \\ \downarrow & & \downarrow \\ \text{Spec } K' & \longrightarrow & X \end{array}$$

commutes. In fact, for any point $x \in X^{\text{an}}$, there is a unique smallest such complete valued field defining x denoted by $\mathcal{H}(x)$ and called the completed residue field of x . We denote by $k(x)$ the residue field of $\mathcal{H}(x)$ and by $\text{val}(x) \subset \mathbf{R}_{>0}$ the group of values of $\mathcal{H}(x)$. Forgetting the valuation, one gets points $\text{Spec}(K) \rightarrow X$ from the same equivalence class of points: this defines a point of X , hence the map $X^{\text{an}} \rightarrow X$. If $U = \text{Spec } A$ is an affine open subset of X , every $x \in \phi^{-1}(U)$ defines a seminorm $|\cdot|_x$ on A . The topology on $\phi^{-1}(U)$ is defined to be the coarsest such that $x \mapsto |f|_x$ is continuous for every $f \in A$.

The space X^{an} is locally compact, and even compact if X is proper. In fact X^{an} is more than just a topological space: it can be enriched into a K -analytic space, as defined by Berkovich in [2].

Points of $\mathbf{A}^{1,\text{an}}$ are of four different types and are described in the following way:

- A closed ball $B = B(a, r) \subset \mathbf{C}_p$ of center a and radius r defines a point $b = b_{a,r}$ of $\mathbf{A}^{1,\text{an}}$ by

$$|f|_b = \sup_{x \in B} |f(x)| = \max_{i \in \mathbf{N}} |a_i| r^i \text{ if } f = \sum_{i \in \mathbf{N}} a_i X^i.$$

The point $b_{a,r}$ is said to be of type 1 if $r = 0$, of type 2 if $r \in p^{\mathbf{Q}}$ and of type 3 otherwise. The pairs (a, r) and (a', r') define the same point if and only if $B(a, r) = B(a', r')$, i.e. $r = r'$ and $|a - a'| \leq r$.

- A decreasing family of balls $E = (B_i)_{i \in I}$ with empty intersection defines a point by

$$|f|_E = \inf_{i \in I} |f|_{b_i}.$$

Such a point is said to be of type 4.

If $r \in p^{\mathbf{Q}}$ and $a \in \mathbf{C}_p$ are such that $|a| = r$, then $|X/a|_{b_{a,r}} = 1$ and $k(b_{a,r}) = k(\overline{X/a})$.

The classification by type of points can be generalized to curves in such a way that it is preserved by finite morphisms: a point x is of type:

- 1 if $\mathcal{H}(x) = \mathbf{C}_p$;
- 2 if $\deg_{\text{tr}} k(x)/\overline{\mathbf{F}}_p = 1$;
- 3 if $\text{val}(x) \neq p^{\mathbf{Q}}$;
- 4 otherwise (*i.e.* $\mathcal{H}(x)/\mathbf{C}_p$ is an immediate extension).

If x is of type 2, we denote by $g_{k(x)}$ the genus of the proper $\overline{\mathbf{F}}_p$ -curve whose field of fraction is $k(x)$.

Let X be a proper and smooth K -curve. The topological space is a quasipolyhedron in the sense of [2, def. 4.1.1]: there exists a base of open subsets U such that:

- $\overline{U} \setminus U$ is finite;
- U is countable at infinity;
- for every $x \neq y \in U$, there exists a unique subset $[x, y] \subset U$ such that homeomorphic to $[0, 1]$ with endpoints x and y .

A quasipolyhedron that satisfies itself these three properties is said to be simply-connected.

If X is a curve, the set of points of type different from 2 is totally disconnected. Therefore, on every nonconstant path, there exists infinitely many points of type 2.

The topological universal cover X^{∞} is a simply-connected quasipolyhedron. Therefore any subset I of X^{∞} is contained in a smallest connected subset $\text{Conv}(I) = \bigcup_{(x,y) \in I^2} [x, y]$. If I, J are closed connected subsets of X^{∞} , one denotes by $[I, J] = \bigcap_{(x,y) \in I \times J} [x, y]$. It is homeomorphic to $[0, 1]$ if $I \cap J = \emptyset$ and $I \cap [I, J]$ and $J \cap [I, J]$ are reduced to a point. If $x \in X^{\infty}$ and I is a closed subset of X^{∞} , one denotes by $r_I(x)$ the unique element of $[x, I] \cap I$. The map $r_I : X^{\infty} \rightarrow I$ is a continuous retraction of the embedding $I \rightarrow X^{\infty}$.

Let \mathcal{X} be a semistable model. There is a specialization map $\pi_{\mathcal{X}} : X^{\text{an}} \rightarrow \mathcal{X}_k$ defined in the following way. If $x \in X^{\text{an}}$, the morphism $\text{Spec } \mathcal{H}(x) \rightarrow X$ extends in a morphism $\text{Spec } \mathcal{O}_{\mathcal{H}(x)} \rightarrow \mathcal{X}$ by properness of $\mathcal{X} \rightarrow \text{Spec } \mathcal{O}_K$, hence a morphism $\text{Spec } k(x) \rightarrow \mathcal{X}_k$: the image of this morphism is $\pi_{\mathcal{X}}(x)$. This specialization map is anticontinuous: the preimage of a closed subset is an open subset.

If z is the generic point of an irreducible component of \mathcal{X}_k , then z is of codimension 1 in \mathcal{X} and thus $\mathcal{O}_{\mathcal{X},z}$ is a valuation ring. The completion of $\text{Frac } \mathcal{O}_{\mathcal{X},z}$ defines a point b_z of X^{an} which is the unique element of $\pi_{\mathcal{X}}^{-1}(z)$. One denotes by $V(\mathcal{X}) \subset X^{\text{an}}$ the set of such b_z and by $V(\mathcal{X})^{\infty}$ its preimage in X^{∞} .

One has $X^{\text{an}} \setminus V(\mathcal{X}) = \coprod \pi_{\mathcal{X}}^{-1}(x)$ where x goes through closed points of \mathcal{X}_k , and $\pi_{\mathcal{X}}^{-1}(x)$ is open by anticontinuity of $\pi_{\mathcal{X}}$. In particular, if $z, z' \in X^{\text{an}}$ are such that $\pi_{\mathcal{X}}(z) \neq \pi_{\mathcal{X}}(z')$, then every path joining z to z' meets $V(\mathcal{X})$.

One denotes by $S(\mathcal{X})^{\infty} = \text{Conv}(V(\mathcal{X})^{\infty})$ and by $r_{\mathcal{X}}^{\infty}$ the retraction $r_{S(\mathcal{X})^{\infty}} : X^{\infty} \rightarrow S(\mathcal{X})^{\infty}$ of the embedding $\iota_{\mathcal{X}}^{\infty} : S(\mathcal{X})^{\infty} \rightarrow X^{\infty}$. Since $V(\mathcal{X})^{\infty}$ is $\text{Gal}(X^{\infty}/X)$ -invariant, $S(\mathcal{X})^{\infty}$ is also $\text{Gal}(X^{\infty}/X)$ -invariant and $r_{\mathcal{X}}^{\infty}$ is $\text{Gal}(X^{\infty}/X)$ -equivariant. One denotes by $S(\mathcal{X})$ the image of $S(\mathcal{X})^{\infty}$ in X^{an} : it is called the skeleton of \mathcal{X} . One denotes by $r_{\mathcal{X}}$ the retraction $X^{\text{an}} \rightarrow S(\mathcal{X})$ induced by $r_{\mathcal{X}}^{\infty}$. The space $S(\mathcal{X})$ is compact and the inclusion map $\iota_{\mathcal{X}} : S(\mathcal{X}) \rightarrow X^{\text{an}}$ is a homotopy equivalence. In fact \mathcal{X} is characterized by the fact that it is the smallest subset S of X^{an} that contains $V(\mathcal{X})$ and such that $S \rightarrow X^{\text{an}}$ is a homotopy equivalence.

If z is a node of \mathcal{X}_k , then $\pi_{\mathcal{X}}^{-1}(z)$ is an open annulus. It contains a unique closed connected subset S_z homeomorphic to \mathbf{R} . More precisely, if one choses an isomorphism of analytic spaces $\pi_{\mathcal{X}}^{-1}(z) \simeq \{z \in \mathbf{A}^{1,\text{an}} | r_0 < |T|_z < 1\}$, then $S_z = \{b_{0,r}, r_0 < r < 1\}$ (in particular the points of type 2 of S_z can be identified

with $\mathbf{Q} \cap (r_0, 1)$). If z is a closed point of the smooth locus of \mathcal{X}_k , then $\pi^{-1}(z)$ is an open disk. Since, for every point b of type 2 of disks and annuli, $k(b)$ is a rational extension of k , one gets that if $x \in X^{\text{an}}$ is a point of type 2 such that $g_{k(x)} \neq 0$, then $x \in V(\mathcal{X})$.

One recovers $S(\mathcal{X})$ as the union of $V(\mathcal{X})$ and of S_z for every node z of \mathcal{X}_k . One gets that $S(\mathcal{X})$ is homeomorphic to the dual graph of \mathcal{X}_k .

If $X_1 \rightarrow X_2$ is a morphism of proper and smooth K -curves and $\mathcal{X}_1 \rightarrow \mathcal{X}_2$ is an extension to semistable O_K -models, then

$$\begin{array}{ccc} X_1^{\text{an}} & \longrightarrow & \mathcal{X}_{1,k} \\ \downarrow & & \downarrow \\ X_2^{\text{an}} & \longrightarrow & \mathcal{X}_{2,k} \end{array}$$

is commutative.

Assume now that $\mathcal{X}_1 \rightarrow \mathcal{X}_2$ is a morphism of semistable models of a same curve X . If z_1 is the generic point of an irreducible component of $\mathcal{X}_{1,k}$ which maps to the generic point z_2 of an irreducible component of $\mathcal{X}_{2,k}$, the previous diagram tells us that $b_{z_1} = b_{z_2}$. Since $\mathcal{X}_1 \rightarrow \mathcal{X}_2$ is surjective, one gets that $V(\mathcal{X}_2) \subset V(\mathcal{X}_1) \subset S(\mathcal{X}_1)$. Since $S(\mathcal{X}_2)$ is the smallest subset of X^{an} that contains $V(\mathcal{X}_2)$ and such that $S(\mathcal{X}_2) \rightarrow X^{\text{an}}$ is a homotopy equivalence, one gets that $S(\mathcal{X}_2) \subset S(\mathcal{X}_1)$. Similarly, one has $V(\mathcal{X}_2)^{\infty} \subset V(\mathcal{X}_1)^{\infty}$ and $S(\mathcal{X}_2)^{\infty} \subset S(\mathcal{X}_1)^{\infty}$.

Therefore, for every $x \in X^{\text{an}}$, $[x, S(\mathcal{X}_1)] \subset [x, S(\mathcal{X}_2)]$, and thus $\iota_{\mathcal{X}_1} r_{\mathcal{X}_1} \in [x, S(\mathcal{X}_2)]$. This implies that $r_{\mathcal{X}_2} \iota_{\mathcal{X}_1} r_{\mathcal{X}_1} = r_{\mathcal{X}_2}$. Therefore, the maps $r_{\mathcal{X}_1/\mathcal{X}_2} := r_{\mathcal{X}_2} \iota_{\mathcal{X}_1} : S(\mathcal{X}_1) \rightarrow S(\mathcal{X}_2)$ are compatible with composition and $(S(\mathcal{X}))_{\mathcal{X}}$ is a projective system of topological spaces. The maps $(r_{\mathcal{X}})$ induce a continuous map

$$r_X : X^{\text{an}} \rightarrow \varprojlim_{\mathcal{X}} S(\mathcal{X}).$$

Proposition 1.1. *The map r_X is a homeomorphism.*

Proof. Since X^{an} is compact, the surjectivity of r_X follows from the surjectivity of each $r_{\mathcal{X}}$.

Let $x \neq x'$ be such that $r_{\mathcal{X}_0}(x) = r_{\mathcal{X}_0}(x')$ where \mathcal{X}_0 is the minimal model of X . Let U be a simply connected open neighborhood of $r_{\mathcal{X}_0}(x)$ in $S(\mathcal{X}_0)$ and $V = r_{\mathcal{X}_0}^{-1}(U)$. Since V is a simply connected quasipolyhedron, there is a minimal connected subset $[x, x'] \subset V$ containing x and x' . It is homeomorphic to $[0, 1]$ and has a natural order that make x the smallest element. Let $x_1 < x_2 \in [x, x']$ be points of type 2.

Since x_i is of type 2, $V \setminus \{x_i\}$ has infinitely many components. Since $\overline{\mathbf{Q}}_p$ -points are dense in X^{an} one can find $z_{i,1}, z_{i,2}, z_{i,3} \in V \cap X(\overline{\mathbf{Q}}_p)$ lying in different connected components of $X^{\text{an}} \setminus \{x_i\}$. Let \mathcal{X} be the stable model of the marked curve $(X, \{z_{ij}\}_{i=1,2;j=1,2,3})$. Since $\pi_{\mathcal{X}}(z_{i,1}) \neq \pi_{\mathcal{X}}(z_{i,2})$, one has $S(\mathcal{X}) \cap [z_{i,1}, z_{i,2}] \neq \emptyset$ and therefore $r_{\mathcal{X}}(z_{i,1}) \in [z_{i,1}, z_{i,2}]$.

Therefore, replacing $z_{i,2}$ by $z_{y,3}$, one gets that $r_{\mathcal{X}}(z_{i,1}) \in [z_{i,1}, z_{i,2}] \cap [z_{i,1}, z_{i,3}] = [z_{i,1}, x_i]$ and similarly $r_{\mathcal{X}}(z_{i,2}) \in [z_{i,2}, x_i]$. Since $S(\mathcal{X})$ is connected and intersects $[z_{i,1}, x_i]$ and $[z_{i,2}, x_i]$, $x_i \in S(\mathcal{X})$. Therefore, in $[x, x']$, $r_{\mathcal{X}}(x) \leq x_1 < x_2 \geq r_{\mathcal{X}}(x')$, which proves the injectivity of r_X .

Since X^{an} is compact and $\varprojlim_{\mathcal{X}} S(\mathcal{X})$ is Hausdorff, r_X is a homeomorphism. \square

Let $\mathcal{X}_1 \rightarrow \mathcal{X}_2$ be a morphism of semistable models of X . Let z be a node of $\mathcal{X}_{2,k}$. We will the notation

$$(1) \quad A_{z, \mathcal{X}_1} := V(\mathcal{X}_1) \cap S_z.$$

If one chooses an orientation of $S_z \simeq \mathbf{R}$, A_{z, \mathcal{X}_1} then becomes an ordered set. One denotes by $A_z = \bigcup_{\mathcal{X}_1} A_{z, \mathcal{X}_1} \subset S_z$.

2. RESOLUTION OF NON-SINGULARITIES

2.1. definition. Let $X = \overline{X} \setminus D$ be a hyperbolic curve over K . Let $X_{(2)}^{\text{an}} \subset X^{\text{an}}$ be the subset of type (2) points.

Let $\tilde{V}(X)$ be the set of points x of X^{an} such that there exists a finite étale cover $f : Y \rightarrow X$ and $y \in V(Y)$ such that $f(y) = x$. If $x \in \tilde{V}(X)$, then Y can be chosen to be Galois, so that in particular $f^{-1}(x) \subset V(Y)$, since $V(Y)$ is Galois equivariant. Let $\overline{V}(X)$ be the closure of $\tilde{V}(X)$ in \overline{X} . If $f : Y \rightarrow X$ is a finite étale cover $\tilde{V}(Y) = f^{-1}(\tilde{V}(X))$ and $\overline{V}(Y) = f^{-1}(\overline{V}(X))$. One has $\tilde{V}(X) \subset X_{(2)}^{\text{an}}$.

Definition 2.1. One says that X satisfies resolution of non-singularities, or $RNS(X)$ for short, if $\tilde{V}(X) = X_{(2)}^{\text{an}}$.

Proposition 2.1. Let $X = \overline{X} \setminus D$ be a curve. The following are equivalent:

- (1) $\tilde{V}(X) = X_{(2)}^{\text{an}}$;
- (2) $\overline{V}(X) = \overline{X}^{\text{an}}$;
- (3) $X(K) \subset \overline{V}(X)$.

Proof. • (i) \Rightarrow (ii). Points of type (2) are dense in X_{an} .
• (ii) \Rightarrow (iii) is obvious.
• (iii) \Rightarrow (i). Let $x \in X_{(2)}$. Then $X^{\text{an}} \setminus \{x\}$ has infinitely many components and they are open in X^{an} . Since $X(K)$ is dense in X^{an} , each of this components intersect $\tilde{V}(X)$. Let $x_1, x_2, x_3 \in \tilde{V}(X)$ lying in different connected components S_1, S_2, S_3 of $X^{\text{an}} \setminus \{x\}$. Let $f : Y \rightarrow X$ be a finite cover such that there is y_i over x_i lying in $S(Y)$ for $i = 1, 2, 3$. Up to replacing Y by a Galois closure, one can assume that $Y \rightarrow X$ is Galois. Since the image T of $S(Y)$ in X^{an} is connected and $x_1 x_2 \in T$, one has $x \in T$. Since $S(Y) \subset Y^{\text{an}}$ is $\text{Gal}(Y/X)$ -invariant, $S(Y) = f^{-1}(T)$. Let $y \in f^{-1}(x)$. For any neighborhood U of x , $U \cap T \cap S_i \neq \emptyset$. Thus for any neighborhood V of y small enough (for example, such that $f^{-1}(x) \cap V = \{y\}$), then $V \cap S(Y) \setminus \{y\}$ has at least three connected components. Thus $y \in V(Y)$ and $x \in \tilde{V}(X)$. \square

If X is a curve over $\overline{\mathbf{Q}}_p$, then $X(\overline{\mathbf{Q}}_p)$ is dense in $X(\overline{\mathbf{C}}_p)$, thus X satisfies resolution of nonsingularities if and only if $X(\overline{\mathbf{Q}}_p) \subset \overline{V}(X)$.

If $Y \rightarrow X$ is a morphism of hyperbolic curves over $\overline{\mathbf{Q}}_p$ and \mathcal{X} is a semistable model of \overline{X} , there exists a minimal semistable model \mathcal{Y} of Y above \mathcal{X} (\mathcal{Y} is the stable marked hull of the normalization of \mathcal{X} in $K(Y)$, in the sense of [10, cor. 2.20]).

Proposition 2.2. Let X be a $\overline{\mathbf{Q}}_p$ -curve which satisfies resolution of nonsingularities. Let \mathcal{X} be a semistable model of \overline{X} and let x be a closed point of \mathcal{X}_k . There exists a finite cover $Y \rightarrow X$ such that \mathcal{Y}_k has a vertical component above x , where \mathcal{Y} is the minimal semistable model of Y above \mathcal{X} .

Proof. Let $\pi_{\mathcal{X}} : \overline{X}^{\text{an}} \rightarrow \mathcal{X}_k$ be the specialization map. Since x is closed, $\pi_{\mathcal{X}}^{-1}(x)$ is open, and therefore contains a point \tilde{x} of type 2. Let Y be a cover of X and let $\tilde{y} \in V(Y)$ be above \tilde{x} . Then, for the stable model \mathcal{Y}_0 of Y , \tilde{y} specializes via $\pi_{\mathcal{Y}_0}$ to a generic point of $\mathcal{Y}_{0,k}$. Therefore, \tilde{y} specializes to a generic point for every semistable model of Y , in particular for \mathcal{Y} . Then $y := \pi_{\mathcal{Y}}(\tilde{y})$ is mapped to x and therefore the closure of y is a vertical component above x . \square

2.2. Splitting points of $\mathbf{Z}_p(1)$ -torsors. The map $\mathbf{G}_m \xrightarrow{(\cdot)^n} \mathbf{G}_m$ defines a μ_n -torsor over \mathbf{G}_m . The corresponding element of $H^1(\mathbf{G}_m, \mu_n)$ is denoted by $c_{\text{can}, n}$.

Let D be a disk centered at 0 and let $f : D \rightarrow \mathbf{G}_m$ be a non constant morphism. Let $c_n = f^* c_{\text{can}, p^n} \in H^1(D, \mu_{p^n})$. Let $Y_n \rightarrow D$ be the corresponding μ_{p^n} -torsor.

Let $f(X) = \sum_{k \geq 0} a_k X^k$ be the power series of f . Let $e_0(f) = \inf\{k \geq 1 | a_k \neq 0\}$ be the ramification index of f at 0.

Let $r_0(c_n) = \inf\{r > 0 | Y_n \text{ is not split above } b_{0,r}\}$ when it exists. Then Y_n is trivial above $D(0, r_0(c_n)^-)$ (otherwise it could be extended in a non trivial finite cover of \mathbf{P}^1).

Let $y_n \in Y_n$ be above $x_n = b_{0, r_0(c_n)}$. The cover $Y_n \rightarrow D$ induces a morphism $\mathcal{H}(x_n) \rightarrow \mathcal{H}(y_n)$ of complete valued field. Let $k(x_n) \rightarrow k(y_n)$ be the morphism of their residue fields.

We want to study the asymptotic behavior of $r_0(c_n)$ when n goes to ∞ and $k(y_n)$.

Proposition 2.3. *There exists C such that, for n big enough, $r_0(c_n) = Cp^{-\frac{n}{e_0(f)}}$.*

Moreover, for n big enough, $[k(y_n) : k(x_n)] = [\mathcal{H}(y_n) : \mathcal{H}(x_n)] = p$ and $k(y_n)$ is isomorphic to $k(X)[T]/(T^p - T - X^{e_0(f)})$.

Proof. Up to multiplying f by a constant, one can assume $f(0) = 1$. Let $N = e_0(f)$. Up to replacing D by a smaller disk, one can assume that $\sum_{k \geq N} a_k X^k$ has a N th root t , so that $f = 1 + t^N$. Moreover up to replacing D by a smaller disk, one can assume that t induces an isomorphism $t : D \rightarrow D_0$ where D_0 is also a disk centered at 0. Since t maps $b_{0,r}$ to $b_{0, \lambda r}$ where λ is a constant, it is enough to prove the result for ft^{-1} . One can thus assume that $f = 1 + X^N$.

One has $f_{n-1} := f^{1/p^{n-1}} = \sum_k b_k X^{Nk}$ where $b_k = \binom{1/p^{n-1}}{k}$ and $v_p(b_k) = -k(n-1) - v_p(k!)$. The series $f^{1/p^{n-1}}$ is convergent on the disk of radius $p^{-(n-1+\frac{1}{p-1})/N}$. By replacing $n-1$ by n , one gets that $r_0(c_n) \geq \lambda_n := p^{-(n+\frac{1}{p-1})/N}$. Let $1+y \in O(Y_n)$ be the p^n th root of f such that $y(0) = 0$. Then y satisfies the equation

$$(2) \quad \sum_{k=1}^p \binom{p}{k} y^k = \sum_{k \geq 1} b_k X^{Nk}.$$

Let $b = b_{0, \lambda_n} \in D$. Let b' be above b in Y_n and let b'' be the image of b' in Y_{n-1} . Since $\lambda_{n-1} > \lambda_n$, the torsor c_{n-1} is split at b'' , one has $[\mathcal{H}(b') : \mathcal{H}(b)] = [\mathcal{H}(b') : \mathcal{H}(b'')]|p$ and $\mathcal{H}(b) = \mathcal{H}(b')/((1+y)^p - f_{n-1})$.

At b , one has $|b_1 X^N|_b = |\frac{1}{p^{n-1}} t^N|_b = p^{-\frac{p}{p-1}}$, and all the other terms in the right member of (2) have smaller norms: $|b_k t^{Nk}|_b = p^{-\frac{kp}{p-1} + v_p(k!)} < p^{-k} \leq p^{-\frac{p}{p-1}}$ for every $k \geq 2$. In particular

$$|\sum_{k=1}^p \binom{p}{k} y^k|_{b'} = |\sum_{k \geq 1} b_k X^{Nk}|_b = p^{-\frac{p}{p-1}}.$$

If $|y|_{b'} < p^{-\frac{1}{p-1}}$, then $|(\binom{p}{k} y^k)|_{b'} = p^{-1} |y^k| < p^{-\frac{p}{p-1}}$ if $1 \leq k < p$ and $|(\binom{p}{k} y^k)|_{b'} = |y|^p < p^{-\frac{1}{p-1}}$ if $k = p$, which is impossible since $|\sum_{k=1}^p \binom{p}{k} y^k|_{b'} = p^{-\frac{p}{p-1}}$. If $|y|_{b'} > p^{-\frac{1}{p-1}}$, then $|(\binom{p}{k} y^k)|_{b'} < |y|^p$ for every $1 \leq k < p$ and therefore $|\sum_{k=1}^p \binom{p}{k} y^k|_{b'} = |y|^p > p^{-\frac{p}{p-1}}$, which is impossible. Therefore, $|y|_{b'} = p^{-\frac{1}{p-1}}$. One gets that $|py| = |y|^p = p^{-\frac{p}{p-1}}$ and $|(\binom{p}{k} y^k)|_{b'} < p^{-\frac{p}{p-1}}$ for every $2 \leq k \leq p-1$.

Therefore, in the ring $\{a \in \mathcal{H}(b') | |a| \leq p^{-\frac{p}{p-1}}\} / \{a \in \mathcal{H}(b') | |a| < p^{-\frac{p}{p-1}}\}$, equation (2) becomes $py + y^p = X^N / p^{n-1}$.

Let $z = y/a_1 \in \mathcal{H}(b')$ and $u = X/a_2 \in \mathcal{H}(b) \subset \mathcal{H}(b')$ where $a_1^{p-1} = -p$ and $a_2^N = p^{n-1} a_1^p$. One has $|z|_{b'} = |u|_{b'} = 1$ and $k(b) = k(\bar{u})$. Let \bar{z} and \bar{u} be the classes

of z and u in $k(b')$. Equation (2) induces in $k(b')$ the equality:

$$\bar{z}^p - \bar{z} = \bar{u}^N.$$

Therefore $k(b)[\bar{z}] \subset k(b')$ is a non trivial extension of $k(b)$. Since $[\mathcal{H}(b') : \mathcal{H}(b)]|p$, one gets that $[\mathcal{H}(b') : \mathcal{H}(b)] = p$, $b = x_n$, and $k(b') = k(b)[\bar{z}]$ is the wanted Artin-Schreier extension of $k(b)$ (because $k(b) = k(\bar{u})$). \square

If one writes $e_0(f) = p^m d$ where d is prime to p , the genus of the Artin-Schreier curve $T^p - T = X^{e_0(f)}$ is $g = (d-1)(p-1)/2$ (cf. [6, §2.2, eq. (8)]). In particular, if $e_0(f)$ is not a power of p , then, for n big enough, $k(y_n)$ is not isomorphic to $k(X)$, so that if Y_n is an analytic domain of a curve \tilde{Y}_n , one must have $y_n \in V(\tilde{Y}_n)$. It should be noticed that $e_0(f) = \text{ord}_0(\frac{df}{f}) + 1$, where $\text{ord}_0 \omega$ denotes the x -adic valuation of $\frac{df}{f} \in \mathbf{C}_p[[X]]$.

2.3. Resolution of non-singularities for Mumford curves. In this subsection we show that Mumford curves over $\overline{\mathbf{Q}}_p$ satisfy resolution of nonsingularities.

A proper curve X over $\overline{\mathbf{Q}}_p$ is a *Mumford curve* if the following equivalent properties are satisfied:

- all normalized irreducible components of its stable reduction are isomorphic to \mathbf{P}^1 ,
- X^{an} is locally isomorphic to $\mathbf{P}^{1,\text{an}}$,

The universal topological covering Ω of X^{an} for a Mumford curve X is an open subset of $\mathbf{P}^{1,\text{an}}$. More precisely there is a Shottky subgroup Γ of $\text{PGL}_2(\mathbf{C}_p)$, *i.e.* a free finitely generated discrete subgroup of $\text{PGL}_2(\mathbf{C}_p)$, such that $\Omega = \mathbf{P}^{1,\text{an}} \setminus \mathcal{L}$ where \mathcal{L} is the closure of the set of \mathbf{C}_p -points stabilized by some nontrivial element of Γ . The points of \mathcal{L} are of type 1, *i.e.* are \mathbf{C}_p -points. Then X is p -adic analytically uniformized as

$$X^{\text{an}} = \Omega/\Gamma$$

and $\Gamma = \pi_1^{\text{top}}(X)$.

Lemma 2.4. *Let I be an infinite subset of K_0 , where $K_0 \subset K$ is a finite extension of \mathbf{Q}_p . Let $x \in \overline{\mathbf{Q}}_p$ be a point not belonging to I . Let $E = \{k \in \mathbf{N} | \exists a \in \mathbf{Q}_p^{(I)}, k = \text{ord}_x(\sum_{i \in I} \frac{a_i}{X-i})\}$ and $u_n = \#(E \cap [0, n])$. Then the sequence $(u_n/n)_n$ does not go to 0 when n goes to infinity.*

Proof. Up to replacing K_0 by $K_0[x]$ and i by $i+x$ for every $i \in I$, one can assume $x = 0$. Let V be the \mathbf{Q}_p -vector subspace of $K_0[[X]]$ generated by $(\frac{1}{X-i})_{i \in I}$. Let $C = [K_0 : \mathbf{Q}_p]$.

Let V_n be the image of the \mathbf{Q}_p -linear map $\phi_n : V \rightarrow K_0[[X]]/X^n$. One has $n \in E$ if and only if $\text{Ker}(\phi_n) \subsetneq \text{Ker}(\phi_{n-1})$. Therefore

- $\dim_{\mathbf{Q}_p} V_n = \dim_{\mathbf{Q}_p} V_{n-1}$ if $n \notin E$;
- $\dim_{\mathbf{Q}_p} V_n \leq C + \dim_{\mathbf{Q}_p} V_{n-1}$.

Therefore $\dim_{\mathbf{Q}_p} V_n \leq C u_n$. However the morphism $f_n : V_n \otimes_{\mathbf{Q}_p} K \rightarrow K[[X]]/X^n$ is surjective. Indeed let i_1, \dots, i_n be n different elements of I and let $\bar{P} \in K[[X]]/X^n$. Let P be a representative of \bar{P} in $K[X]$. Let R be the remainder of $P \prod_{k=1}^n (X - i_k)$ by the division by X^n . Then, since $\deg(R) < n$, $\frac{R}{\prod_{k=1}^n (X - i_k)} = \sum_k \frac{a_k}{X - i_k}$ for some $(a_k)_k$ in K^n , and, since $\prod_k (X - i_k)$ is invertible in $K[[X]]$, $\bar{P} = f_n(\sum_k \frac{1}{X - i_k} \otimes a_i)$, which proves that f_n is surjective. Therefore $\dim_{\mathbf{Q}_p} V_n = \dim_{\mathbf{K}} V_n \otimes_{\mathbf{Q}_p} K \geq n$ and $u_n/n \geq 1/C$. \square

Let I be an infinite subset of K_0 , where $K_0 \subset K$ is a finite extension of \mathbf{Q}_p . Let $x \in \overline{\mathbf{Q}_p}$. Lemma 2.3 shows that there exists $(a_i) \in \mathbf{Q}_p^{(I)}$ be such that $g = \sum \frac{a_i}{x-i}$ is such that $\text{ord}_x(g) + 1$ is not a power of p . Up to multiplying all the a_i by $p^{\max_{i \in I} (-v_p(a_i))}$, one can assume $a_i \in \mathbf{Z}_p$. Let $I_0 \subset I$ be the support of the family $(a_i)_{i \in I}$.

For $i \in I_0$ and $n \geq 0$, let $a_{i,n} \in \mathbf{Z}$ be such that $v_p(a_{i,n} - a_i) \geq n$.

Let $f_n = \prod_{i \in I_0} (\frac{x-i}{x-i})^{a_{i,n}} : \mathbf{P}^1 \setminus I_0 \rightarrow \mathbf{G}_m$. Let $D = \{z \in \mathbf{G}_m \mid |X - x|_z < \min_{i \in I_0} |x - i|\}$. The sequence (f_n) is uniformly convergent on every affinoid subset of D and defines over D a morphism $f : D \rightarrow \mathbf{G}_m$ and $f'/f = g$ over D .

Let $c_n : Y_n \rightarrow \mathbf{P}^1 \setminus I_0$ be the μ_{p^n} -torsor over $\mathbf{P}^1 \setminus I_0$ obtained by pulling back along f_n the canonical torsor (remark that c_n only depends on a_i and not on $a_{i,n}$). The restriction of c_n to D is also the pullback of the canonical torsor along f .

According to proposition 2.3, there is a point $y_n \in Y_n$ such that $g_{k(y_n)} \geq 1$ and $c_n(y_n) \rightarrow x$.

One gets the following result:

Proposition 2.5. *Let I be an infinite subset of a finite extension of \mathbf{Q}_p . For every $x \in \overline{\mathbf{Q}_p}$, there is a finite subset $I_0 \subset I \setminus \{x\}$ and a $\mathbf{Z}_p(1)$ -torsor $c = (c_n : Y_n \rightarrow \mathbf{P}^1 \setminus I_0)$ of $\mathbf{P}^1 \setminus I_0$ and for every $n \geq 1$ a point $y_n \in Y_n$ such that $g_{k(y_n)} \geq 1$ and $c_n(y_n) \rightarrow x$.*

Theorem 2.6. *Let X be a Mumford curve over $\overline{\mathbf{Q}_p}$. Then, X satisfies resolution of non-singularities.*

Proof. Let $x \in X(\overline{\mathbf{Q}_p})$. According to proposition 2.1, it is enough to show that $x \in \overline{V}(X)$.

Let $\Omega = \mathbf{P}^{1,\text{an}} \setminus \mathcal{L}$ be the topological universal cover of X , and let $\Gamma = \text{Gal}(\Omega/X) \subset \text{PGL}_2(\mathbf{C}_p)$. Let z be a point of Ω above x . One can assume that $\Gamma \subset \text{PGL}_2(K_0)$, where K_0 is a finite extension of \mathbf{Q}_p . Let $g \in \Gamma$. Let $t \in \mathcal{L} \cap \mathbf{P}^1(\overline{\mathbf{Q}_p})$ be a point that is not fixed by g . Up to replacing K_0 by a finite extension, one can assume $t \in K_0$. Let $I = \{g^n(t)\}_{n \in \mathbf{Z}} \subset K_0$. According to proposition 2.5, there exists a finite subset I_0 of I and a $\mathbf{Z}_p(1)$ -torsor $(c_n : Y_n \rightarrow \mathbf{P}^1 \setminus I_0)$ of $\mathbf{P}^1 \setminus I_0$ and a point y_n of Y_n such that $g_{k_{y_n}} \geq 1$ and $z_n := c_n(y_n) \rightarrow z$. Let x_n be the image of z_n in X . Fix n and show that $x_n \in \widetilde{V}(X)$. Let ϵ be small enough so that the canonical μ_{p^n} torsor of \mathbf{G}_m is split at $b_{1,\epsilon}$. Let $f \in O(\mathbf{P}^1 \setminus I_0)^*$ such that $c_n = f^* \mu_{p^n}$.

Let $z_0 \in \Omega(\mathbf{C}_p)$. By replacing f by $f(z)/f(z_0)$, one can assume $f(z_0) = 1$. Let Γ' be a subgroup of Γ of finite index. Consider $f_{\Gamma'}(z) = \prod_{g \in \Gamma'} \frac{f(g(z))}{f(g(z_0))}$: this product converges uniformly on every affinoid domain of Ω and therefore defines an element of $O(\Omega)$. Moreover, for every $g \in \Gamma'$, $\frac{f_{\Gamma'} \circ g}{f_{\Gamma'}}$ is a constant function, i.e. $f_{\Gamma'}$ is a theta function of X/Γ' . If $(\Gamma_m)_{m \in \mathbf{N}}$ is a decreasing sequence of subgroups of finite index of Γ such that $\bigcap_m \Gamma_m = \{1\}$, the sequence $(f_{\Gamma_m})_{m \in \mathbf{N}}$ converges uniformly to f on every affinoid domain of Ω . In particular, there exists a subgroup Γ' of finite index of Γ such that $|f_{\Gamma'}/f - 1|_{z_n} < \epsilon$. Let $c' = f_{\Gamma'}^* \mu_{p^n} : Y' \rightarrow \Omega$. Then $c_n - c'$ is split at z_n . Therefore c_n and c' are isomorphic above z_n and there is $y' \in Y'$ above z_n such that $g_{k_{y'_n}} \geq 1$. Since $f_{\Gamma'}$ is a theta function of Ω/Γ' , there exists a μ_{p^n} torsor $c'' : Y'' \rightarrow \Omega/\Gamma'$ such that $c' = p^* c''$ where p is the topological cover $\Omega \rightarrow \Omega/\Gamma'$. Since $Y' \rightarrow Y''$ is a topological cover, the image y'' of y' in Y'' is in $V(Y'')$. Since $Y'' \rightarrow X$ is a finite cover and maps y'' to x_n , $x_n \in \widetilde{V}(X)$. Therefore $x \in \overline{V}(X)$. \square

Since $\mathbf{P}^1 \setminus \{0, 1, \infty\}$ and punctured Tate curves have finite étale covers that are nonempty Zariski open subsets of Mumford curves, they also satisfy resolution of nonsingularities. For every curve X over \mathbf{C}_p , there exists a nonempty Zariski open subset $U \subset X$ and a finite cover $U \rightarrow \mathbf{P}^1 \setminus \{x_1, \dots, x_n\}$ with $n \geq 3$. Since

$\mathbf{P}^1 \setminus \{x_1, \dots, x_n\}$ satisfies resolution of non singularities, U also satisfy resolution of nonsingularities:

Corollary 2.7. *Every \mathbf{C}_p -curve has a Zariski dense open subset which satisfies resolution of nonsingularities.*

3. RESOLUTION OF NON SINGULARITIES AND ANABELIAN TEMPERED GEOMETRY

3.1. Tempered fundamental group. Let K be a complete nonarchimedean field.

A morphism $f : S' \rightarrow S$ of K -analytic spaces is said to be an *étale cover* if S is covered by open subsets U such that $f^{-1}(U) = \coprod V_j$ and $V_j \rightarrow U$ is étale finite ([5]).

For example, finite étale covers, also called *algebraic covers*, and covers in the usual topological sense for the Berkovich topology, also called *topological covers*, are étale covers.

Then, André defines tempered covers as follows:

Definition 3.1. ([1, def. 2.1.1]) An étale cover $S' \rightarrow S$ is *tempered* if it is a quotient of the composition of a topological cover $T' \rightarrow T$ and of a finite étale cover $T \rightarrow S$.

This is equivalent to say that it becomes a topological cover after pullback by some finite étale cover.

We denote by $\text{Cov}^{\text{temp}}(X)$ (resp. $\text{Cov}^{\text{alg}}(X)$, $\text{Cov}^{\text{top}}(X)$) the category of tempered covers (resp. algebraic covers, topological covers) of X (with the obvious morphisms).

A geometric point of a K -manifold X is a morphism of Berkovich spaces $\mathcal{M}(\Omega) \rightarrow X$ where Ω is an algebraically closed complete isometric extension of K .

Let \bar{x} be a geometric point of X . Then one has a functor

$$F_{\bar{x}} : \text{Cov}^{\text{temp}}(X) \rightarrow \text{Set}$$

which maps a cover $S \rightarrow X$ to the set $S_{\bar{x}}$. If \bar{x} and \bar{x}' are two geometric points, then $F_{\bar{x}}$ and $F_{\bar{x}'}$ are (non canonically) isomorphic ([5, prop. 2.9]).

A functor $F : \text{Cov}^{\text{temp}}(X) \rightarrow \text{Set}$ is said to be a *fiber functor* if it is isomorphic to $F_{\bar{x}}$ for some (and therefore every) geometric point \bar{x} of X .

Proposition 3.1. *A fiber functor of $\text{Cov}^{\text{temp}}(X)$ is pro-representable.*

If F is a fiber functor of $\text{Cov}^{\text{temp}}(X)$, a pointed tempered cover of X is a couple (S, s) where S is a tempered cover of X and $s \in F(S)$. A pro-tempered cover \tilde{X} of X is called universal if $F_{\tilde{X}} := \text{Hom}(\tilde{X}, \)$ is a fiber functor of $\text{Cov}^{\text{temp}}(X)$.

The tempered fundamental group of X pointed at a fiber functor F is

$$\pi_1^{\text{temp}}(X, F) = \text{Aut } F.$$

The tempered fundametal group of X pointed at a universal pro-tempered cover \tilde{X} is

$$\pi_1^{\text{temp}}(X, \tilde{X}) = \text{Aut } \tilde{X} = \text{Aut } F_{\tilde{X}}.$$

The tempered fundamental group of X pointed at a geometric point \bar{x} is

$$\pi_1^{\text{temp}}(X, \bar{x}) = \pi_1^{\text{temp}}(X, F_{\bar{x}})$$

When X is a smooth algebraic K -variety, $\text{Cov}^{\text{temp}}(X^{\text{an}})$ and $\pi_1^{\text{temp}}(X^{\text{an}}, F)$ will also be denoted simply by $\text{Cov}^{\text{temp}}(X)$ and $\pi_1^{\text{temp}}(X, F)$.

By considering the stabilizers $(\text{Stab}_{F(S)}(s))_{(S,s)}$ as a basis of open subgroups of $\pi_1^{\text{temp}}(X, F)$, $\pi_1^{\text{temp}}(X, F)$ becomes a topological group. It is a prodiscrete topological group.

When X is algebraic, K of characteristic zero and has only countably many finite extensions in a fixed algebraic closure \bar{K} , $\pi_1^{\text{temp}}(X, F)$ has a countable fundamental

system of neighborhood of 1 and all its discrete quotient groups are finitely generated ([1, prop. 2.1.7]).

Thus, as usual, the tempered fundamental group depends on the basepoint only up to inner automorphism (this topological group, considered up to conjugation, will sometimes be denoted simply by $\pi_1^{\text{temp}}(X)$). The full subcategory of tempered covers S for which $F_{\bar{x}}(S)$ is finite is equivalent to $\text{Cov}^{\text{alg}}(S)$, hence

$$\widehat{\pi_1^{\text{temp}}(X, \bar{x})} = \pi_1^{\text{alg}}(X, \bar{x})$$

(where $\widehat{}$ denotes here, and in the sequel, the profinite completion).

For any morphism $X \rightarrow Y$, the pullback defines a functor $\text{Cov}^{\text{temp}}(Y) \rightarrow \text{Cov}^{\text{temp}}(X)$. If \bar{x} is a geometric point of X with image \bar{y} in Y , this gives rise to a continuous homomorphism

$$\pi_1^{\text{temp}}(X, \bar{x}) \rightarrow \pi_1^{\text{temp}}(Y, \bar{y})$$

(hence an outer morphism $\pi_1^{\text{temp}}(X) \rightarrow \pi_1^{\text{temp}}(Y)$).

One has the analog of the usual Galois correspondence:

Theorem 3.2. ([1, th. 1.4.5]) *A fiber functor F induces an equivalence of categories between the category tempered covers of X and the category $\pi_1^{\text{temp}}(X, F)$ -Set of discrete sets endowed with an action of $\pi_1^{\text{temp}}(X, F)$ through factorizes through a finite quotient.*

If S is a finite Galois cover of X , its universal topological cover S^∞ is still Galois and every connected tempered cover is dominated by such a Galois tempered cover. Let \bar{x} be a geometric point of X . Let $\varprojlim_I (S_i, s_i)$ be a pointed universal profinite cover of X . Let (S_i^∞, s_i^∞) be the pointed universal topological cover of (S_i, s_i) . Then $\varprojlim_I S_i^\infty$ is a universal pro-tempered cover of X .

If $S = \varprojlim S_i$ is a pro-tempered cover of X , one denotes by $|S|$ the topological space $\varprojlim S_i$.

Let (\bar{X}, D) be a marked curve. Let $X = \bar{X} \setminus D$. If S is a tempered cover of X , it extends uniquely, and functorially in Y , in a ramified cover $\bar{S} \rightarrow \bar{X}$ ([1, th. III.2.1.11]). If $S = \varprojlim S_i$ is a pro-tempered cover of X , one denotes by $|S|_c$ the topological space $\varprojlim \bar{S}_i$. Any morphism $S \rightarrow S'$ of pro-tempered covers induces a continuous map $|S|_c \rightarrow |S'|_c$. In particular, if \tilde{X} is a universal pro-tempered cover of X , $\pi_1^{\text{temp}}(X, \tilde{X}) = \text{Aut } \tilde{X}$ acts on $|\tilde{X}|_c$.

Let \mathcal{P} be the set of prime numbers and Let \mathbb{L} be a subset of \mathcal{P} . We call \mathbb{L} -integer an integer which is a product of elements of \mathbb{L} . One writes (p') for $\mathcal{P} \setminus \{p\}$.

A \mathbb{L} -tempered cover S of X is a tempered cover such that there exists a finite étale Galois cover $Y \rightarrow X$ of index a \mathbb{L} -integer such that $S \times_X Y \rightarrow Y$ is a topological cover. We denote by $\text{Cov}^{\text{temp}}(X)^{\mathbb{L}}$ the category of \mathbb{L} -tempered covers. If F is a fiber functor of $\text{Cov}^{\text{temp}}(X)$, we denote by $\pi_1^{\text{temp}}(X, F)^{\mathbb{L}}$ the topological group of automorphisms of $F|_{\text{Cov}^{\text{temp}}(X)^{\mathbb{L}}}$. If \tilde{X} is a universal pro-tempered cover of X , one defines $\tilde{X}^{\mathbb{L}} = \varprojlim_{\tilde{X} \rightarrow Y} Y$ where Y runs over \tilde{X} -pointed \mathbb{L} -tempered cover.

If $\mathbb{L} \subset \mathbb{L}'$, the fully faithful functor $\text{Cov}^{\text{temp}}(X)^{\mathbb{L}} \rightarrow \text{Cov}^{\text{temp}}(X)^{\mathbb{L}'}$ induces a morphism $\pi_1^{\text{temp}}(X, F)^{\mathbb{L}'} \rightarrow \pi_1^{\text{temp}}(X, F)^{\mathbb{L}}$, and in particular when $\mathbb{L}' = \mathcal{P}$, one gets a morphism $\pi_1^{\text{temp}}(X, F) \rightarrow \pi_1^{\text{temp}}(X, F)^{\mathbb{L}}$.

3.2. Decomposition group of a point. Let (\bar{X}, D) be a marked K -curve, and let $X = \bar{X} \setminus D$. Let \tilde{X} be a universal pro-tempered cover. One defines $\tilde{X}^{\mathbb{L}} = \varprojlim_Y Y^\infty$ where Y runs over the \tilde{X} -pointed finite étale covers of X of index an

\mathbb{L} -integer and Y^∞ is the \tilde{X} -pointed universal topological cover of Y . If $p \notin \mathbb{L}$, every morphism $Z \rightarrow Y$ of \tilde{X} -pointed finite étale covers of X of index an \mathbb{L} -integer induces a morphism of tree $\mathbb{T}_Z \rightarrow \mathbb{T}_Y$; one defines $\mathbb{T}^\mathbb{L}$ to be $\varinjlim_Y \mathbb{T}_Y$. The family of embeddings $V(\mathbb{T}_Y) \rightarrow Y^\infty$ induces an embedding $V(\mathbb{T}_X^\mathbb{L}) \rightarrow |\tilde{X}^\mathbb{L}|$. If $z = (z_Y)$ is an edge of $\mathbb{T}_X^\mathbb{L}$, then every morphism $Y \rightarrow Y'$ induces a homeomorphism $S_{z_Y} \rightarrow S_{z'_Y}$ and one defines $S_z = \varprojlim_Y S_{z_Y}$.

Then $\pi_1^{\text{temp}}(X, \tilde{X})$ acts on $|\tilde{X}|_c$. Similarly $\pi_1^{\text{temp}}(X, \tilde{X})^\mathbb{L}$ acts on $|\tilde{X}^\mathbb{L}|_c$. Let $x \in |\tilde{X}|_c$. One denotes by D_x the stabilizer of x in $\pi_1^{\text{temp}}(X, \tilde{X})$, and by $D_{x, \mathbb{L}}$ its image in $\pi_1^{\text{temp}}(X, \tilde{X})^\mathbb{L}$. The group $D_{x, \mathbb{L}}$ is the stabilizer of the image of x in $|\tilde{X}^\mathbb{L}|_c$. The decomposition group depends only of the image of x in \overline{X} up to conjugacy.

Theorem 3.3 ([11, cor. 3.11]). *If X_α and X_β are two hyperbolic $\overline{\mathbb{Q}}_p$ -curves, every (outer) isomorphism $\gamma : \pi_1^{\text{temp}}(X_\alpha, \mathbb{C}_p)^{(p')} \simeq \pi_1^{\text{temp}}(X_\beta, \mathbb{C}_p)^{(p')}$ determines, functorially in γ , an isomorphism of graphs $\bar{\gamma} : \mathbb{G}_{X_\alpha} \simeq \mathbb{G}_{X_\beta}$.*

More precisely, the map $x \mapsto D_{x, (p')}$ identifies the vertices of \mathbb{T}_X with the maximal compact subgroups of $\pi_1^{\text{temp}}(X)^{(p')}$ and two vertices x, x' of \mathbb{T}_X are linked by an edge if and only if $D_{x, (p')} \cap D_{x', (p')} \neq \{1\}$. Therefore the isomorphism $\gamma : \pi_1^{\text{temp}}(X_\alpha, \mathbb{C}_p)^{(p')} \simeq \pi_1^{\text{temp}}(X_\beta, \mathbb{C}_p)^{(p')}$ induces an equivariant isomorphism of graphs $\mathbb{T}_{X_\alpha} \rightarrow \mathbb{T}_{X_\beta}$, which gives $\bar{\gamma}$ by quotienting by the action of the tempered fundamental group.

Proposition 3.4. *Let $x_1 \neq x_2 \in \tilde{V}(\tilde{X})$. Then D_{x_1} and D_{x_2} are not commensurable.*

Proof. Let $(Y, f : \tilde{X} \rightarrow Y)$ be a pointed finite étale cover of X such that $f(x_1) \neq f(x_2) \in V(Y)$. Then $D_{x_i} \cap \pi_1^{\text{temp}}(Y, \tilde{X}) = D_{f(x_i)}$. But the images of $D_{f(x_1)}$ and $D_{f(x_2)}$ in $\pi_1^{\text{temp}}(Y, \tilde{X})^{(p')}$ are already not commensurable. \square

Corollary 3.5. *Let $x \in \tilde{V}(\tilde{X})$. Then D_x is its own normalizer.*

Proof. Let g be in the normalizer of D_x . Then $D_{g(x)} = gD_xg^{-1} = D_x$. Since $x, g(x) \in \tilde{V}(\tilde{X})$, proposition 3.4 tells us that $x = g(x)$, i.e. $g \in D_x$. \square

According to [7, prop. 10], if D is a compact subgroup of $\pi_1^{\text{temp}}(X, \overline{X})$ which is not a pro- p group, there exists $x \in |\tilde{X}|_c$ such that $D \subset D_x$. Moreover, a point $x \in |\tilde{X}|_c$ is in $\tilde{V}(\tilde{X})$ if and only if there exists an open finite index subgroup $H \subset \pi_1^{\text{temp}}(X)$ such that the image of $D_x \cap H$ in $H^{(p')}$ is non commutative. Therefore, the set $\tilde{V}(\tilde{X})$ can be identified with the set of conjugacy classes of maximal compact subgroups D of $\pi_1^{\text{temp}}(X)$ such that the image of $D \cap H$ in $H^{(p')}$ is non commutative for some open finite index subgroup H of $\pi_1^{\text{temp}}(X)$.

3.3. Tempered theoreticness of Berkovich topology. Let (\overline{X}, D) be a $\overline{\mathbb{Q}}_p$ -marked curve and let $X = \overline{X} \setminus D$. If $Y \rightarrow X$ is a Galois finite étale cover and $(\overline{Y}, \mathcal{D}_Y)$ is the stable model of Y , $\mathcal{X}_Y := \overline{Y} / \text{Gal}(Y/X)$ is a semistable model of \overline{X} .

Let us say that a topological group is temp-like if it is isomorphic to the tempered fundamental group of a hyperbolic curve over $\overline{\mathbb{Q}}_p$ that satisfies resolution of non-singularities. We will construct for any temp-like topological group Π a topological space $\tilde{S}(\Pi)$ endowed with a continuous action of Π . The construction will be purely group theoretic, so that it will be functorial with respect to isomorphism of topological groups. Moreover when $\Pi = \pi_1^{\text{temp}}(X, \tilde{X})$, we will get a $\pi_1^{\text{temp}}(X, \tilde{X})$ -equivariant homeomorphism $|\tilde{X}| \rightarrow \tilde{S}(\pi_1^{\text{temp}}(X, \tilde{X}))$.

Let Π be a temp-like topological group. We fix an isomorphism $\Pi \simeq \pi_1^{\text{temp}}(X, \tilde{X})$ (but we will take care that the construction of $\tilde{S}(\Pi)$ do not depend on this isomorphism). Then there is a smallest normal open subgroup Π^∞ such that Π/Π^∞ is torsionfree. One can also see Π^∞ as the closed subgroup generated by the compact subgroups of Π . One defines the topological group $\Pi^{(p')} := \varprojlim_N \Pi/N^\infty$ where N goes through open normal subgroups of finite index prime to p of Π (such an N is also temp-like, so that N^∞ is well defined). The morphism $\Pi \rightarrow \Pi^{(p')}$ has dense image. We denote by $\Pi^{(p'),\infty}$ the kernel of $\Pi^{(p')} \rightarrow \Pi/\Pi^\infty$.

If H is a normal open subgroup of Π of finite index, let $\tilde{V}(\Pi)_H$ be the set of maximal compact subgroups D of Π such that $D \cap H$ is not commutative. Let $\tilde{V}(\Pi) = \bigcup_H \tilde{V}(\Pi)_H$. The group Π acts by conjugacy on $\tilde{V}(\Pi)$ and on $\tilde{V}(\Pi)_H$ for every H .

Recall that there is an equivariant bijection $\tilde{V}(\tilde{X}) \rightarrow \tilde{V}(\Pi)$ that maps x to D_x . More precisely, If $(Y, f : \tilde{X} \rightarrow Y)$ is the pointed finite étale cover of X corresponding to H , it induces a bijection $\{x \in |\tilde{X}| : f(x) \in V(Y)\} \rightarrow \tilde{V}(\Pi)_H$. By quotienting by H^∞ , one gets a bijection $V(Y)^\infty \rightarrow \tilde{V}(\Pi)_H/H^\infty$.

Let $H \subset \Pi$ be a normal open subgroup of finite index, H is also temp-like: it is isomorphic to $\pi_1^{\text{temp}}(Y, \tilde{X})$ for some Galois finite étale cover Y of X . Let $V(H)^{(p')}$ be the set of maximal compact subgroups of $H^{(p')}$. Let $E(H)^{(p')}$ be the set of pairs of elements (D, D') of $V(H)^{(p')}$ such that $D \cap D' \neq \{1\}$. These data define a graph $\mathbb{G}(H)^{(p')}$. Since H is normal in Π , the group Π acts on $H^{(p')}$ by conjugacy and therefore on $\mathbb{G}(H)^{(p')}$. Remark that $H^{(p')}$ also acts by conjugacy on $\mathbb{G}(H)^{(p')}$ and that the action of Π and of $H^{(p')}$ coincide on H .

According to [7, Th. 6], there is a Π -equivariant isomorphism $\mathbb{T}_Y^{(p')} \simeq \mathbb{G}(H)^{(p')}$ that maps a vertex x to its stabilizer D_x by the action of $H^{(p')}$. If $e = (D, D') \in E(H)^{(p')}$, one denotes by $D_e := D \cap D' \subset H^{(p')}$. The group D_e is the stabilizer of the image of e in $\mathbb{T}_Y^{(p')}$ for the action of $H^{(p')}$ on $\mathbb{G}(H)^{(p')}$.

Let

$$\mathbb{G}(H) = \mathbb{G}(H)^{(p')}/H^{(p')} \quad \text{and} \quad \mathbb{G}(H)^\infty = \mathbb{G}(H)^{(p')}/H^{(p'),\infty}.$$

Then $\mathbb{G}(H)$ can be identified with \mathbb{G}_Y and $\mathbb{G}(H)^\infty$ with \mathbb{T}_Y .

If $D \in \tilde{V}(\Pi)_H$, the image of $D \cap H$ in $H^{(p')}$ is a maximal compact subgroup, and therefore defines an element of $V(H)^{(p')}$, hence a Π -equivariant map $p_H : \tilde{V}(\Pi)_H \rightarrow V(H)^{(p')}$.

Moreover, the induced map $p_H^\infty : \tilde{V}(\Pi)_H/H^\infty \rightarrow V(H)^{(p')}/H^{(p'),\infty}$ is bijective: Indeed, the diagram

$$\begin{array}{ccc} \tilde{V}(\Pi)_H/H^\infty & \longrightarrow & V(H)^{(p')}/H^{(p'),\infty} \\ & \searrow & \downarrow \\ & & V(Y^\infty) \end{array}$$

is commutative and the two vertical maps are bijections.

One has $H^\infty \subset \Pi^\infty$ and one gets a Π -equivariant map $\iota_{H,\Pi}^{(p')} : H^{(p')} \rightarrow \Pi^{(p')}$.

Lemma 3.6. *Let $D \in V(H)^{(p')}$. The subgroup $\iota_{H,\Pi}^{(p')}(D) \subset \Pi^{(p')}$ is either:*

- an open subgroup of D_1 for a unique $D_1 \in V(\Pi)^{(p')}$;
- an open subgroup of D_e for a unique $e \in \overline{E}(\Pi)^{(p')}$;
- $\{1\}$.

Let $e \in E(H)^{(p')}$. The subgroup $\iota_{H,\Pi}^{(p')}(D_e) \subset \Pi^{(p')}$ is either:

- an open subgroup of D_e for a unique $e \in \overline{E}(\Pi)^{(p')}$;
- $\{1\}$.

Proof. Uniqueness is clear.

Since p_H^∞ is surjective, up to conjugating D by an element of $H^{(p')}$, one can assume that D is the image of p_H . Let $D_0 \in \tilde{V}(\Pi)_H$ be a preimage of D and let x be the corresponding point of $\tilde{V}(\tilde{X})$. Since $H \cap D_x$ is open in D_x , $\iota_{H,\Pi}^{(p')}(D)$ is open in the image $D_{x,(p')}$ of D_x by $\Pi \rightarrow \Pi^{(p')}$. Let \tilde{x} be the image of x in $\tilde{X}^{(p')}$. If $\tilde{x} \in V(\tilde{X}^{(p')})$, then $D_{x,(p')} \in V(\Pi)^{(p')}$; if \tilde{x} lies in an edge e of $\mathbb{T}_X^{(p')}$, then $D_{x,(p')} = D_e$; otherwise, the image of x in X lies in a disk and since every prime-to- p cover of a disk is trivial, $D_{x,(p')} = \{1\}$.

if $e \in E(H)^{(p')}$, there exists $\bar{y} \in |\tilde{Y}^{(p')}|$ such that $D_e = D_x$. Up to conjugating D by an element of $H^{(p')}$, one can assume that there is $x \in |\tilde{X}|$ which maps to \bar{y} . Let \bar{x} be the image of \bar{y} in $|\tilde{X}^{(p')}|$. Once again, $\iota_{H,\Pi}^{(p')}(D_e)$ is open in $D_{\bar{x}}$. Since $\bar{y} \notin V(Y)^{(p')}$, $\bar{x} \notin V(X)^{(p')}$. Therefore either \bar{x} lies in an edge e' of $\mathbb{T}_X^{(p')}$ and $D_{\bar{x}} = D_{e'}$ or x lies outside the image of $S(X)^{(p')}$ and $D_{\bar{x}} = \{1\}$. \square

One denotes by $\mathbb{G}_{\Pi^\infty}(H) = \mathbb{G}(H)^\infty/\Pi^\infty$. If one identifies $\mathbb{G}(H)^\infty$ with \mathbb{T}_Y , then $\mathbb{G}_{\Pi^\infty}(H) = \mathbb{T}(\mathcal{X}_Y)$. The bijection $p_H^{\infty,-1}$ induces a bijection $V_{\Pi^\infty}(H) \rightarrow \tilde{V}(\Pi)_H/\Pi^\infty$. If $H' \subset H$ are two normal subgroups of Π , the Π -equivariant injective map $\tilde{V}(\Pi)_H \rightarrow \tilde{V}(\Pi)_{H'}$ induces an injective map $V_{\Pi^\infty}(H) \rightarrow V_{\Pi^\infty}(H')$.

Let $e = (D_1, D_2) \in E(\Pi)^\infty$. One denotes by

$$\tilde{A}_{e,H}^{(p')} := \{D \in V(H)^{(p')} \mid \exists \tilde{e} \in E(\Pi)^{(p')}, [\tilde{e}] = e \text{ and } \{1\} \neq \iota_{H,\Pi}^{(p')}(D) \subset D_{\tilde{e}}\}$$

The action of $H^{(p'),\infty}$ and of Π^∞ on $V(H)^{(p')}$ stabilize $\tilde{A}_{e,H}$. Let

$$A_{e,H} = (\tilde{A}_{e,H}^{(p')}/H^{(p'),\infty})/\Pi^\infty$$

Thus $A_{e,H}$ is a subset of $V_{\Pi^\infty}(H)$. An element $D \subset \Pi$ of $\tilde{V}(\Pi)_H$ is mapped to $A_{e,H}$ by $\tilde{V}(\Pi)_H \rightarrow V_{\Pi^\infty}$ if and only if the image of D in $\Pi^{(p')}$ is a representative of e . If $Y \rightarrow X$ is the pointed Galois cover corresponding to H , then $A_{e,H}$ can be identified with $A_{z,Y}$, as defined in (1) where z is the node of \mathcal{X}_Y corresponding to e .

The full subgraph $\mathbb{G}(A_{e,H})$ of $\mathbb{G}_{\Pi^\infty}(H)$ with vertices $A_{e,H} \cup \{i_{H,\Pi}(D_1), i_{H,\Pi}(D_2)\}$ is a line (indeed the embedding $|\mathbb{G}(A_{e,H})| \subset |\mathbb{T}_{\mathcal{X}_Y}| \subset X^\infty$ identifies $|\mathbb{G}(A_{e,H})|$ with $\overline{S}_z \simeq [0, 1]$ where z is the edge of \mathbb{T}_X corresponding to e), so that $A_{e,H}$ is naturally a totally ordered set for which D_1 is the minimal element and D_2 is the maximal. If $\bar{e} = [(D_2, D_1)] \in E(\Pi)^\infty$ is the same edge of $\mathbb{G}(\Pi)^\infty$ with the opposite orientation, then there is an obvious bijection $A_{e,H} \simeq A_{\bar{e},H}$, which is decreasing.

Let $H' \subset H \subset \Pi$ be two finite index normal subgroup of Π , the injective map $V_{\Pi^\infty}(H) \rightarrow V_{\Pi^\infty}(H')$ maps $A_{e,H}$ to $A_{e,H'}$. The induced map $A_{e,H} \rightarrow A_{e,H'}$ is increasing. Let

$$A_e := \varinjlim_H A_{e,H},$$

where H goes through finite index normal subgroups of Π . By identifying $A_{e,H}$ with $A_{z,Y}$, one gets an increasing bijection $A_e \simeq A_z$. Since X satisfies resolution of non-singularities, A_z can be identified with the set of points of type 2 of S_z . Thus A_e is an ordered set which is, non-canonically, isomorphic to $\mathbf{Q} \cap (0, 1)$. Let \hat{A}_e be the Dedekind completion of A_e : \hat{A}_e is an ordered topological space non-canonically isomorphic to $[0, 1]$. The decreasing bijections $A_{e,H} \rightarrow A_{\bar{e},H}$ are compatible and

therefore induce a homeomorphism $\phi_e : \widehat{A}_e \rightarrow \widehat{A}_{\bar{e}}$. Let us call 0_e (resp. 1_e) the minimal element of \widehat{A}_e . One then defines the topological space

$$S(\Pi)^\infty := \left(V(\Pi)^\infty \coprod_{e \in E(\Pi)^\infty} \widehat{A}_e \right) / \sim$$

where \sim is generated by

$$\forall (D, D') \in E(\Pi)^\infty, D \sim 0_{(D, D')}, \quad \forall e \in E(\Pi)^\infty \forall x \in \widehat{A}_e, x \sim \phi_e(x).$$

Since \widehat{A}_e is non-canonically homeomorphic to $[0, 1]$, $S(\Pi)^\infty$ is non-canonically homeomorphic to the geometric realization of $\mathbb{G}(\Pi)^\infty$.

If H is a finite index open normal subgroup of Π , one similarly gets a topological space $S(H)^\infty$ and the action of Π on H by conjugacy induces an action of Π on $S(H)^\infty$.

Let $\tilde{e} \in E(H)^{(p')}$ and $\tilde{e}_0 \in E(\Pi)^{(p')}$ be such that $\iota_{H, \Pi}^{(p')}(\tilde{e})$ is an open subgroup of $D_{\tilde{e}_0}$. Let $H' \subset H$ be an open normal subgroup of Π . One has $\tilde{A}_{e, H'} \subset \tilde{A}_{e_0, H'}$, as subsets of $V(H')^{(p')}$. Hence a map

$$A_{e, H'} := \tilde{A}_{e, H'} / H \rightarrow A_{e_0, H'} := A_{e_0, H'}.$$

Since open normal subgroups of Π which are inside H are cofinal among open normal subgroups of Π and among normal subgroups of H , one gets by taking colimits a map

$$\alpha_{e, e_0} : A_e \rightarrow A_{e_0}$$

Lemma 3.7. *There exists at most one continuous map $\psi_{H, \Pi} : S(H)^\infty \rightarrow S(\Pi)^\infty$ such that:*

- (i) *if $e \in E(H)^\infty$ and $\iota_{H, \Pi}^{(p')}(\tilde{e}) = 1$, then $\psi_{H, \Pi}$ is constant on \widehat{A}_e ;*
- (ii) *if $\tilde{e} \in E(H)^{(p')}$ and $\tilde{e}_0 \in E(\Pi)^{(p')}$ are such that $\iota_{H, \Pi}^{(p')}(\tilde{e})$ is an open subgroup of $D_{\tilde{e}_0}$, $\psi_{H, \Pi}|_{A_e} = \alpha_{e, e_0}$.*

Proof. Assume ψ and ψ' satisfy the condition. For every $\tilde{e} \in E(H)^{(p')}$ such that $\iota_{H, \Pi}^{(p')}(\tilde{e}) \neq 1$, then thanks to (ii), $\psi = \psi'$ on A_e and therefore on \widehat{A}_e . For every $\tilde{v} \in V(H)^{(p')}$ such that $\iota_{H, \Pi}^{(p')}(\tilde{v}) \neq 1$, there exists $\tilde{e} \in E(H)^{(p')}$ ending at \tilde{v} such that $\iota_{H, \Pi}^{(p')}(\tilde{v}) \neq 1$, and therefore $\psi(v) = \psi'(v)$. Since $\mathbb{G}(\Pi)^\infty$ is connected, one can link every edge and vertex of $\mathbb{G}(\Pi)^\infty$ by a finite path to a vertex such that $\iota_{H, \Pi}^{(p')}(\tilde{v}) \neq 1$. Up to reducing the path one can assume that for every edge e of the path $\iota_{H, \Pi}^{(p')}(\tilde{e}) = 1$. By induction on the length of such a path, one gets that $\psi = \psi'$. using (i). \square

There is a Π -equivariant injection $V(\Pi)^{(p')} \rightarrow X^{(p')}$ that maps D to the unique $x \in X^{(p')}$ such that $D = D_x$. By quotienting by Π^∞ , one gets a Π -equivariant injection $V(\Pi)^\infty \rightarrow X^\infty$. Similarly, for every finite index subgroup H of Π , there is a Π -equivariant injection $V(H)^\infty \rightarrow X^\infty$, which induces by quotienting by Π^∞ an injection $V_{\Pi^\infty}(H) \rightarrow X^\infty$. If $H \subset H'$ are two finite index normal subgroups, the following diagram is commutative:

$$\begin{array}{ccc} V_{\Pi^\infty}(H) & \longrightarrow & V_{\Pi^\infty}(H') \\ & \searrow & \downarrow \\ & & X^\infty \end{array}$$

Let e be an edge of $\mathbb{G}(\Pi)^\infty$, and let z be the corresponding node of \mathcal{X}_k . One gets compatible injective maps $A_{e,H} \rightarrow X^\infty$, whose image lie in S_z , hence an injective map $f_e : A_e \rightarrow S_z$, compatible with the reversing of edges. Since A_e is dense in \widehat{A}_e , there is at most one extension of f_e into a map $\widehat{A}_e \rightarrow X^\infty$, necessarily compatible with edges. Since X satisfies resolution of non-singularities, the image of A_e is exactly the set of points of type (2) in S_z . However, if \mathcal{X} is isomorphic to $\text{Spec } O_K[X, Y]/(XY - a)$ in an étale neighborhood of z , then \overline{S}_z can be identified with $[0, v(a)]$, and the set of points of type (2) of S_z is $\mathbf{Q} \cap (0, v(a))$. On $A_{e,H}$ identified with a finite subset of $[0, v(a)]$, the tree structure is simply given by joining the consecutive points. Therefore $A_e \rightarrow [0, v(a)]$ is monotonous. Therefore f_e extends to a unique homeomorphism $\tilde{f}_e : \widehat{A}_e \rightarrow S_z$, that preserves the end-points and is compatible with the reversing of edges. By gluing these maps, one therefore gets a continuous bijection $f^\infty : S(\Pi)^\infty \rightarrow S(X^\infty)$, which is a homeomorphism since $S(\Pi)^\infty$ is locally compact.

Similarly, if H is a finite index open subgroup of Π , one gets a Π -equivariant homeomorphism $S(H)^\infty \rightarrow S(Y^\infty)$.

The composition $S(H)^\infty \simeq S(Y^\infty) \xrightarrow{r_X f^\infty \iota_Y} S(X^\infty) \simeq S(\Pi)^\infty$, where f^∞ is the map $Y^\infty \rightarrow X^\infty$, satisfies the properties of lemma 3.7. Therefore there exists a unique map $S(H)^\infty \simeq S(\Pi)^\infty$ satisfying the properties of lemma 3.7.

If $H' \subset H$ are two finite index open subgroups of Π , then, since H is also temp-like, there exists a unique map $S(H')^\infty \rightarrow S(H)^\infty$ satisfying the properties of lemma 3.7 and this map is Π -equivariant by uniqueness. If $H'' \subset H'$, then the diagram

$$\begin{array}{ccc} S(H'')^\infty & \longrightarrow & S(H')^\infty \\ & \searrow & \downarrow \\ & & S(H)^\infty \end{array}$$

is commutative. One therefore gets a projective system $(S(H)^\infty)$ and one defines:

$$\tilde{S}(\Pi) = \varprojlim_H S(H)^\infty.$$

The equivariant homeomorphisms induce a Π -equivariant homeomorphism $S(H)^\infty \rightarrow S(Y^\infty)$ induce a Π -equivariant homeomorphism

$$\tilde{S}(\Pi) \rightarrow \varprojlim_Y S(Y^\infty).$$

The maps $|\tilde{X}|_c \rightarrow \overline{Y}^\infty \xrightarrow{r_Y} S(Y^\infty)$ are compatible and therefore induce a Π -equivariant map $|\tilde{X}|_c \rightarrow \varprojlim_Y S(Y^\infty)$.

Lemma 3.8. *The map $|\tilde{X}|_c \rightarrow \varprojlim_Y S(Y^\infty)$ is a homeomorphism.*

Proof. First let us show that the map $r : \overline{X}^{\text{an}} = |\tilde{X}|_c/\Pi \rightarrow \varprojlim_Y S(Y^\infty)/\Pi = \varprojlim_Y S(\mathcal{X}_Y)$ is a homomorphism (the proof is similar to the proof of prop. 1.1). Since $r_{\mathcal{X}_Y} : \overline{X}^{\text{an}} \rightarrow S(\mathcal{X}_Y)$ is surjective for every Y and \overline{X}^{an} is compact, r is surjective. Let $x_1 \neq x_2 \in \overline{X}^{\text{an}}$ and show that $r(x_1) \neq r(x_2)$. One can assume $r_X(x_1) = r_X(x_2)$. Let $[x_1, x_2]$ be the smallest subset of X containing x_1 and x_2 , endowed with the total order such that $x_1 < x_2$. Let $y_1 < y_2$ be two points of type 2 in $[x_1, x_2]$ and let $Y \rightarrow X$ be a finite Galois cover such that $y_1, y_2 \in V(\mathcal{X}_Y)$. Then $r_{\mathcal{X}_Y}(x_1) < y_1 < y_2 < r_{\mathcal{X}_Y}(x_2)$ in $[x_1, x_2]$, which proves the injectivity of $r_{\mathcal{X}_Y}$ and therefore of r . Since \overline{X}^{an} is compact, r is an homeomorphism.

By pulling back along $S(X^\infty) \rightarrow S(X)$, one gets that

$$r^\infty : \overline{X}^\infty = \overline{X}^{\text{an}} \times_{S(X)} S(X^\infty) \rightarrow \varprojlim_Y S(\mathcal{X}_Y) \times_{S(X)} S(X^\infty) = \varprojlim_Y S(\mathcal{X}_Y^\infty)$$

is a homeomorphism.

Similarly one gets that the map $\overline{Y}^\infty \rightarrow \varprojlim_Z S(\mathcal{Y}_Z^\infty)$, where Z goes through Galois pointed cover of Y (one can even restrict to Z over Y Galois over X since they are cofinal among Galois pointed cover of Y) is a homeomorphism for every Y Galois. Therefore the map

$$|\tilde{X}|_c \rightarrow \varprojlim_{Z \rightarrow Y \rightarrow X} S(\mathcal{Y}_Z^\infty) \rightarrow \varprojlim_Z S(Z^\infty)$$

is a homeomorphism (the right arrow is a homeomorphism because the full subcategory of the category of morphisms $Y \rightarrow Z$ of pointed Galois cover over X which consists of isomorphisms is a cofinal category). \square

One thus gets an equivariant homeomorphism

$$(3) \quad \tilde{S}(\Pi) \rightarrow |\tilde{X}|_c.$$

Let (\overline{X}, D) be a $\overline{\mathbb{Q}}_p$ -marked curve and let $X = \overline{X} \setminus D$. If $Y \rightarrow X$ is a Galois finite étale cover and $(\overline{Y}, \mathcal{D}_Y)$ is the stable model of Y , $\mathcal{X}_Y := \overline{Y} / \text{Gal}(Y/X)$ is a semistable model of \overline{X} and one gets a refinement $\phi_Y : \mathbb{G}_{Y/X} := \mathbb{G}_Y / \text{Gal}(Y/X) = \mathbb{G}_{\mathcal{Y}_s / \text{Gal}(Y/X)} \rightarrow \mathbb{G}_X$.

If X satisfies non resolution of singularities, the family $(\overline{Y} / \text{Gal}(Y/X))_Y$ is cofinal among semistable models of \overline{X} . Thus, if $e \in \mathcal{E}(X)$, $A_e = \varinjlim A_{\phi_Y, e}$ and $\overline{X}^{\text{an}} \rightarrow \varprojlim_Y |\mathbb{G}_{Y/X}|_{\text{can}}$ is a homeomorphism.

Theorem 3.9. *Let $X_1 = \overline{X}_1 \setminus D_1, X_2 = \overline{X}_2 \setminus D_2$ be two marked curves satisfying resolution of non singularities and let \tilde{X}_i be a universal pro-tempered cover of X_i . Let $\psi : \pi_1^{\text{temp}}(X_1, \tilde{X}_1) \simeq \pi_1^{\text{temp}}(X_2, \tilde{X}_2)$ be an isomorphism. Then there exists a unique homeomorphism $\bar{\psi} : |\tilde{X}_1|_c \rightarrow |\tilde{X}_2|_c$ which is π_1^{temp} -equivariant in the sense that the following diagram commutes:*

$$\begin{array}{ccc} \pi_1^{\text{temp}}(X_1, \tilde{X}_1) \times |\tilde{X}_1|_c & \longrightarrow & |\tilde{X}_1|_c \\ \downarrow \psi \times \bar{\psi} & & \downarrow \bar{\psi} \\ \pi_1^{\text{temp}}(X_2, \tilde{X}_2) \times |\tilde{X}_2|_c & \longrightarrow & |\tilde{X}_2|_c \end{array}$$

In particular, by quotienting by the tempered fundamental group, one gets a homeomorphism:

$$\overline{X}_1^{\text{an}} \rightarrow \overline{X}_2^{\text{an}}$$

Proof. First, assume $\bar{\psi}_1, \bar{\psi}_2 : |\tilde{X}_1|_c \rightarrow |\tilde{X}_2|_c$ are two π_1^{temp} -equivariant homeomorphisms. Then, if $x \in |\tilde{X}_1|_c$, $\psi(D_x) = D_{\bar{\psi}_1(x)} = D_{\bar{\psi}_2(x)}$. If x is in $\tilde{V}(\tilde{X}_1)$, then there exists an open subgroup H of finite index of $\pi_1^{\text{temp}}(X_1, \tilde{X}_1)$ such that $(D_x \cap H)^{(p')}$ is not commutative. Then $\psi(H)$ is a finite index subgroup of $\pi_1^{\text{temp}}(X_2, \tilde{X}_2)$ and $(D_{\bar{\psi}_1(x)} \cap \psi(H))^{(p')}$ and $(D_{\bar{\psi}_2(x)} \cap \psi(H))^{(p')}$ are not commutative. Therefore $\bar{\psi}_1(x)$ and $\bar{\psi}_2(x)$ are of type 2 and have the same decomposition group: according to proposition 3.4, $\bar{\psi}_1(x) = \bar{\psi}_2(x)$ for every point of $\tilde{V}(\tilde{X}_1)$. Since $\tilde{V}(\tilde{X}_1)$ is dense in $|\tilde{X}_1|_c$, one has $\bar{\psi}_1 = \bar{\psi}_2$.

The morphism ψ induces an equivariant homeomorphism $\tilde{S}(\pi_1^{\text{temp}}(X_1, \tilde{X}_1)) \rightarrow \tilde{S}(\pi_1^{\text{temp}}(X_2, \tilde{X}_2))$. One gets from (3) equivariant homeomorphisms $|\tilde{X}_1|_c \rightarrow \tilde{S}(\pi_1^{\text{temp}}(X_1, \tilde{X}_1))$ and $|\tilde{X}_2|_c \rightarrow \tilde{S}(\pi_1^{\text{temp}}(X_2, \tilde{X}_2))$. One gets the wanted isomorphism by composition.

□

Proposition 3.10. *Let $\psi_a, \psi_b : \pi_1^{\text{temp}}(X_1, \tilde{X}_1) \rightarrow \pi_1^{\text{temp}}(X_2, \tilde{X}_2)$ be two isomorphisms. If $\tilde{\psi}_a = \tilde{\psi}_b$, then $\psi_a = \psi_b$.*

Proof. Let $g \in \pi_1^{\text{temp}}(X_1, \tilde{X}_1)$. If $x_1 \in |\tilde{X}_1|$, then

$$\psi_a(g) D_{\tilde{\psi}_a(x_1)} \psi_a(g)^{-1} = D_{\tilde{\psi}_a(gx_1)} = D_{\tilde{\psi}_b(x_1)} = \psi_b(g) D_{\tilde{\psi}_b(x_1)} \psi_b(g)^{-1} = \psi_b(g) D_{\tilde{\psi}_a(x_1)} \psi_b(g)^{-1}$$

which implies that $g_0 := \psi_a(g)^{-1} \psi_b(g)$ is in the normalizer $N_{\psi_a(x_1)}$ of $D_{\psi_a(x_1)}$. Since ψ_a is bijective, $g_0 \in \bigcap_{x_2 \in \tilde{X}_2} N_{x_2}$. If $x_2 \in \tilde{V}(\tilde{X}_2)$, then $N_{x_2} = D_{x_2}$. Therefore $g_0(x_2) = x_2$ for every $x_2 \in \tilde{V}(\tilde{X}_2)$. Since X_2 satisfies resolution of non singularities, $\tilde{V}(\tilde{X}_2)$ is dense in \tilde{X}_2 , and thus $g_0(x) = x$ for every $x \in \tilde{X}_2$. If x is of type 1, then $D_x = \{1\}$. Therefore $g_0 = 1$, i.e. $\psi_a(g) = \psi_b(g)$. □

In particular, if (X, \tilde{X}) is a pointed curve satisfying resolution of non singularities, one has an injective morphism of groups: $\text{Aut } \pi_1^{\text{temp}}(X, \tilde{X}) \rightarrow \text{Aut } |\tilde{X}|$.

4. TATE CURVES

Let $q_1, q_2 \in \overline{\mathbf{Q}_p}$ such that $|q_1| < 1$ and $|q_2| < 1$. Let $E_i = \mathbf{G}_m / q_i^{\mathbf{Z}}$ and let $X_i = E_i \setminus \{1\}$.

If there exists $\sigma \in G_{\mathbf{Q}_p}$ such that $q_1 = \sigma(q_2)$, there is a \mathbf{C}_p -isomorphism $X_1 \simeq X_2 \otimes_{\mathbf{C}_p} \mathbf{C}_p$ where the base change $\mathbf{C}_p \rightarrow \mathbf{C}_p$ is σ . Therefore X_1 and X_2 are isomorphic analytic spaces over \mathbf{Q}_p and therefore have isomorphic tempered fundamental group. The following theorem states that the converse is also true:

Theorem 4.1. *Let $\psi : \pi_1^{\text{temp}}(X_1, \tilde{X}_1) \simeq \pi_1^{\text{temp}}(X_2, \tilde{X}_2)$ be an isomorphism. There exists $\sigma \in G_{\mathbf{Q}_p}$ such that $q_1 = \sigma(q_2)$, i.e. E_1 and E_2 are isomorphic analytic spaces over \mathbf{Q}_p .*

Remark. The curves X_1 and X_2 satisfy the assumptions of theorem 3.9, and thus ψ induces a homeomorphism $E_1^{\text{an}} \rightarrow E_2^{\text{an}}$. However, the author does not know, even in this situation, if this homeomorphism comes from an analytic morphism. The author does not know how to associate to ψ a particular σ .

Proof. Recall that $|q_1| = |q_2|$ and that ψ induces a unique equivariant homeomorphism $\tilde{\psi} : |\tilde{X}_1|_c \simeq |\tilde{X}_2|_c$. The induced homeomorphism $E_1 \simeq E_2$ maps X_1 onto X_2 . Let Ω_i be the universal \tilde{X}_i -pointed topological cover of X_i and let $\tilde{\psi} : |\Omega_1| \rightarrow |\Omega_2|$ the homeomorphism induced by $\tilde{\psi}$.

Let $E_{i,l}$ be the unique \tilde{X}_i -pointed connected topological cover of E_i of degree l . Let $X_{i,l} = X_i \times_{E_i} E_{i,l}$. The isomorphism ψ induces an isomorphism $\psi_l : \pi_1^{\text{temp}}(X_{1,l}, \tilde{X}_1) \rightarrow \pi_1^{\text{temp}}(X_{2,l}, \tilde{X}_2)$, whence an isomorphism

$$\psi_{l,n} : H^1(X_{2,l}, \mu_n) \rightarrow H^1(X_{1,l}, \mu_n)$$

functorial in l for divisibility.

Let \mathbb{G}_i be the semigraph of X_i and let \mathbb{T}_i be its universal cover. Let g be the isomorphism $\mathbb{T}_1 \rightarrow \mathbb{T}_2$ induced by ψ . One identifies $\overline{\Omega}_i$ with \mathbf{G}_m in such a way that $\bar{\psi}(1) = 1$ and $\bar{\psi}(q_1) = q_2$. If $j \in \mathbf{Z}$, one denotes by $e_{i,j}$ the cuspidal edge of \mathbb{T}_i corresponding to $q_i^j \in \overline{\Omega}_i \setminus \Omega_i$ and by $v_{i,j}$ the vertex of \mathbb{T}_i at which $e_{i,j}$ ends. Since $\bar{\psi}(1) = 1$ and $\bar{\psi}(q_1) = q_2$, $g(e_{1,j}) = e_{2,j}$ and $g(v_{1,j}) = v_{2,j}$. One denotes by $e'_{i,j}$ the unique oriented edge joining $v_{i,j}$ to $v_{i,j+1}$.

All the cohomology groups will be cohomology groups for étale cohomology in the sense of algebraic geometry or in the sense of Berkovich. (one can replace étale

cohomology of X^{an} by étale cohomology of X thanks to [4, thm. 3.1]). Since $\Gamma_l \simeq \mathbf{Z}$, one has $H^2(\Gamma, \mu_n) = 0$. Therefore the spectral sequence

$$H^p(\Gamma_l, H^q(\Omega, \mu_n)) \implies H^n(X_{i,l}, \mu_n)$$

of the Galois étale cover $\Omega_i \rightarrow X_{i,l}$ gives us an exact sequence of cohomology groups for Berkovich étale topology:

$$1 \rightarrow \text{Hom}(\Gamma_l, \mu_n) \rightarrow H^1(X_{i,l}, \mu_n) \rightarrow H^1(\Omega_i, \mu_n)^{\Gamma_l} \rightarrow 1.$$

The map $O^*(\Omega_i)/O^*(\Omega_i^n) \rightarrow H^1(\Omega_i, \mu_n)$ given by Kummer theory (see [3, prop. 4.1.7] for the Kummer exact sequence in Berkovich étale topology) induces an morphism

$$(O^*(\Omega_i)/O^*(\Omega_i^n))^{\Gamma_l} \rightarrow H^1(\Omega_i, \mu_n)^{\Gamma_l},$$

which turns to be an isomorphism (cf. [8, §1.4.2]); hence an exact sequence

$$1 \rightarrow \text{Hom}(\Gamma_l, \mu_n) \rightarrow H^1(X_{i,l}, \mu_n) \rightarrow (O^*(\Omega_i)/O^*(\Omega_i^n))^{\Gamma_l} \rightarrow 1.$$

where $\Gamma_l = \text{Gal}(\Omega_i/X_{i,l})$.

One can describe $O^*(\Omega_i)$ in terms of currents (as done in [14] for Mumford curves). If A is a ring, a A -current on \mathbb{T}_i is a function $c : \{e_{i,j}\}_{j \in \mathbf{Z}} \coprod \{e'_{i,j}\}_{j \in \mathbf{Z}} \rightarrow A$ such that for every $j \in \mathbf{Z}$, $c(e'_{i,j+1}) = c(e'_{i,j}) + c(e_{i,j+1})$. Let $C(\mathbb{T}_i, A)$ be the A -module of A -currents on \mathbb{T}_i . There is a natural isomorphism $\alpha_i : C(\mathbb{T}_i, \mathbf{Z}) \rightarrow O^*(\Omega_i)/C_p^*$ defined by

$$\alpha_i(c) = x^{c(e'_{i,0})} \prod_{j \geq 1} \left(\frac{x - q_i^j}{x} \right)^{c(e_{i,j})} \prod_{j \leq 0} \left(\frac{x - q_i^j}{q_i^j} \right)^{c(e_{i,j})}$$

Conversely, if $f \in O^*(\Omega_i)$, one can compute $\alpha_i^{-1}(f)$ in the following way. For every $j \in \mathbf{Z}$, the restriction of f to the open annulus $U_j = \{z \in \mathbf{P}^{1,\text{an}}, |q_i^j| < |x|_z < |q_i^{j+1}|\}$ can be written in a unique way as $f(x) = x^{m_j} g_j(x)$ where $m_j \in \mathbf{Z}$ and $|g_j|$ is constant on U_j . One has $\alpha_i^{-1}(f)(e'_{i,j}) = m_j$. Similarly, the restriction of f to the punctured open disk $V_j = \{z \in \mathbf{P}^{1,\text{an}}, 0 < |x - q_i^j|_z < |q_i^j|\}$ can be written in a unique way as $f(x) = x^{n_j} f_j(x)$ where $n_j \in \mathbf{Z}$ and $|f_j|$ is constant on V_j . One has $\alpha_i^{-1}(f)(e_{i,j}) = n_j$.

Therefore, one gets an isomorphism $\alpha_{i,n} : O^*(\Omega_i)/O^*(\Omega_i)^n \rightarrow C(\mathbb{T}_i, \mathbf{Z}/n\mathbf{Z})$, hence an exact sequence:

$$1 \rightarrow \text{Hom}(\Gamma_l, \mu_n) \rightarrow H^1(X_{i,l}, \mu_n) \rightarrow C(\mathbb{T}_i, \mathbf{Z}/n\mathbf{Z})^{\Gamma_l} \rightarrow 1.$$

Since $\varinjlim_l \text{Hom}(\Gamma_l, \mu_n) = 0$, it induces an isomorphism

$$a_i : \varinjlim_l H^1(X_{i,l}, \mu_n) \rightarrow C(\mathbb{T}_i, \mathbf{Z}/n\mathbf{Z})^{(\Gamma)},$$

where $C(\mathbb{T}_i, \mathbf{Z}/n\mathbf{Z})^{(\Gamma)}$ is the set of $\mathbf{Z}/n\mathbf{Z}$ -currents on \mathbb{T}_i that are invariant under some finite index subgroup of Γ .

Consider the diagram:

$$\begin{array}{ccc} \varinjlim_l H^1(X_{2,l}, \mu_n) & \xrightarrow{f := \varinjlim_l \psi_{l,n}} & \varinjlim_l H^1(X_{1,l}, \mu_n) \\ \downarrow a_1 & & \downarrow a_2 \\ C(\mathbb{T}_2, \mathbf{Z}/n\mathbf{Z})^{(\Gamma)} & \xrightarrow{g^*} & C(\mathbb{T}_1, \mathbf{Z}/n\mathbf{Z})^{(\Gamma)} \end{array}$$

where the lower arrow is induced by $g : \mathbb{T}_1 \rightarrow \mathbb{T}_2$.

The following lemma shows that this diagram is commutative up to a constant in $(\mathbf{Z}/n\mathbf{Z})^*$ (cf. [8, prop. 13] for a similar result for Mumford curves of genus greater than 2).

Proposition 4.2. *There exists a unique $\alpha \in (\mathbf{Z}/n\mathbf{Z})^*$ such that $a_2 f = \alpha g^* a_1$.*

Proof. Let $\tilde{f} = a_2 f a_1^{-1}$. We have to show that there exists $\lambda \in (\mathbf{Z}/n\mathbf{Z})^*$ such that for every $c \in C(\mathbb{T}_2, \mathbb{Z}/n\mathbb{Z})^{(\Gamma)}$ and every edge e of \mathbb{T}_1 , $\tilde{f}(c)(e) = \lambda c(g(e))$. Let $j \in \mathbf{Z}$. According to [9, lem. 4.2], a finite cover of $X_{1,l}$ is ramified at $e_{1,j}$ if and only if the corresponding cover of $X_{2,l}$ is unramified at $e_{2,j}$. Therefore $\tilde{f}(c)(e_{1,j}) = 0$ if and only if $c(e_{2,j}) = 0$. Therefore there exists $\lambda_j \in (\mathbf{Z}/n\mathbf{Z})^*$ such that $\tilde{f}(c)(e_{1,j}) = \lambda_j c(e_{2,j})$ for every $c \in C(\mathbb{T}_2, \mathbb{Z}/n\mathbb{Z})^{(\Gamma)}$. Let l be a positive integer. Let $j \in \mathbf{Z}$ and let c_j be the Γ_l -invariant current defined by

$$c_j(e_{1,k}) = \begin{cases} 1 & \text{if } k = j \pmod{l} \\ -1 & \text{if } k = j+1 \pmod{l} \\ 0 & \text{otherwise} \end{cases}$$

$$c_j(e'_{1,k}) = \begin{cases} 0 & \text{if } k \in [j+1, j+l-1] \pmod{l} \\ 1 & \text{if } k = j \pmod{l} \end{cases}$$

If l is big enough (for example if $l \geq 2 + 2\frac{v_p(n)+2}{|q_1|}$ according to [9, cor. 4.10]), the μ_n -torsor corresponding to c_j is split at $v_{1,j+\lceil \frac{l+1}{2} \rceil}$. Therefore, the μ_n -torsor corresponding to $\tilde{f}(c_j)$ is split at $v_{2,j+\lceil \frac{l+1}{2} \rceil}$, which implies, according to [9, prop. 4.11], that $\tilde{f}(c_j)$ is zero at all the edges ending at $v_{2,j+\lceil \frac{l+1}{2} \rceil}$. Then $\tilde{f}(c_j)(e_{2,j}) = \lambda_j$, $\tilde{f}(c_j)(e_{2,j+1}) = \lambda_{j+1}$, $\tilde{f}(c_j)(e_{2,k}) = 0$ for all $k \neq j, j+1 \pmod{l}$, and is zero for some non cuspidal edge between $v_{2,j+1}$ and $v_{2,j+l}$. Therefore $\tilde{f}(c_j) = \lambda_j g^*(c_j)$ and $\lambda_j = \lambda_{j+1}$. One thus gets that λ_j does not depend on j , one simply denotes it by λ . The group of Γ_l -equivariant current is generated by $(c_j)_{j \in [0, 2l-1]}$ so that one gets $\tilde{f}(c) = \lambda g^*(c)$ for every current c , which ends the proof. \square

Let A_i be the multiplicative group of non-zero meromorphic functions on \mathbf{G}_m with no poles and no zeroes on $\Omega_i \subset \mathbf{G}_m$. Let $B_i \subset A_i$ be the subgroup of A_i consisting of functions for which 1 is neither a pole nor a zero. Let $A'_i \subset \Omega^1(\Omega_i)$ be the groupe of regular differentials on \mathbf{G}_m with no poles on Ω_i .

The map $d \log \circ \alpha_i : C(\mathbb{T}_i, \mathbf{Z}) \rightarrow A'_i$ can be extended by linearity to a map $\delta_i : C(\mathbb{T}_i, \mathbf{Z}_p) \rightarrow A'_i$ defined by

$$\delta_i(c) = c(e'_{i,0}) \frac{dx}{x} + \sum_{j \geq 1} c(e_{i,j}) \left(\frac{dx}{x - q_i^j} - \frac{dx}{x} \right) + \sum_{j \leq 0} c(e_{i,j}) \frac{dx}{x - q_i^j}.$$

If $z \in \mathbf{G}_m(\mathbf{C}_p)$, one denotes by $\omega \in A'_i$, one denotes by $\text{ord}_z(\omega) \in \mathbf{Z}_{\geq -1}$ the $(x - z)$ -adic valuation of $\frac{\omega}{dx} \in \frac{1}{x-z} \mathbf{C}_p[[x-z]]$.

Lemma 4.3. *Let c be in $C(\mathbb{T}_1, \mathbf{Z}_p)$, and let $z \in \mathbf{G}_m(\mathbf{C}_p)$. Assume that $\bar{\psi}(z)$ is also of type 1, i.e. $\bar{\psi}(z) \in \mathbf{G}_m(\mathbf{C}_p)$. Then $\text{ord}_z(\delta_1(c)) = \text{ord}_{\bar{\psi}(z)}(\delta_2(g^*(c)))$.*

Proof. Let $J = \{j \in \mathbf{Z} | C(e_{1,j}) \neq 0\}$. For $z = q_1^j$ with $j \in J$, then $\bar{\psi}(q_1^j) = q_2^j$ and thus $\text{ord}_{q_1^j}(\delta_1(c)) = \text{ord}_{q_2^j}(\delta_2(g^*(c))) = -1$. One can thus assume $z \in \mathbf{G}_m(\mathbf{C}_p) \setminus \{q_1^j\}_{j \in J}$ and therefore $\text{ord}_z(\delta_1(c)), \text{ord}_{\bar{\psi}(z)}(\delta_2(g^*(c))) \geq 0$.

Let $c_n \in C(\mathbb{T}_1, \mathbf{Z})^{(\Gamma)}$ be such that $|c_n(e_{1,j}) - c(e_{1,j})|, |c_n(e'_{1,j}) - c(e'_{1,j})| \leq p^{-n}$ if $-|j|_\infty \log_p |q_1| \leq n + \frac{p}{p-1}$.

Then $\delta_1(c_n) \rightarrow \delta_1(c)$ on every affinoid subspace of $\mathbf{G}_m \setminus \{q_1^j\}_{j \in J}$. Similarly $\delta_2(g^*(c_n)) \rightarrow \delta_2(g^*(c))$ on every affinoid subspace of $\mathbf{G}_m \setminus \{q_2^j\}_{j \in J}$. Therefore

$$\text{ord}_z(\delta_1(c)) \geq m \iff \forall \epsilon > 0 \exists N \forall n \geq N, \sum_{z' \in D(z, \epsilon)} \text{ord}_{z'}(\delta_1(c_n)) \geq m.$$

Therefore it is enough to prove the lemma for every c_n . One thus assumes that $c \in C(\mathbb{T}_1, \mathbf{Z})^{(\Gamma)}$. Let N be such that $c \in C(\mathbb{T}_1, \mathbf{Z})^{\Gamma_N}$. The isomorphism ψ induces an isomorphism $\psi_N : \pi_1^{\text{temp}}(X_{1,N}) \rightarrow \pi_1^{\text{temp}}(X_{2,N})$.

Consider the image Y_1 of c under the map

$$C(\mathbb{T}_1, \mathbf{Z}) \rightarrow O^*(\Omega_1) \rightarrow H^1(\Omega_1, \mathbf{Z}_p(1)),$$

and let $Y_{1,n}$ be the induced μ_{p^n} -torsor on Ω_1 ; it extends in an unramified μ_{p^n} -torsor of $\Omega_1 \setminus \{q_1^j\}_{j \in J}$. Let $z_{1,n}$ be the point of Ω_1 such that $Y_{1,n}$ is not split at $z_{1,n}$ but is split on $[z, z_{1,n})$. Consider the image Y_2 of $g^*(c)$ under the map $C(\mathbb{T}_2, \mathbf{Z}) \rightarrow O^*(\Omega_2) \rightarrow H^1(\Omega_2, \mathbf{Z}_p(1))$, let $Y_{2,n}$ be the induced μ_{p^n} -torsor on Ω_2 and let $z_{2,n}$ be the point of Ω_2 such that $Y_{2,n}$ is not split at $z_{2,n}$ but is split on $[\bar{\psi}(z), z_{2,n})$. Since c is Γ_N -invariant, $Y_{i,n}$ is in the image of $\text{Hom}(\pi_1^{\text{temp}}(X_{i,N}), \mu_{p^n}) = H^1(X_{i,N}, \mu_{p^n}) \rightarrow H^1(\Omega_i, \mu_{p^n})$. For every preimage β_i of $Y_{i,n}$ in $\text{Hom}(\pi_1^{\text{temp}}(X_{i,N}), \mu_{p^n})$, for every $Y_{i,n}$ is split at a point $z' \in \mathbf{G}_m$ if and only if $D_{z'} \subset \text{Ker}(\beta_i)$, where $D_{z'}$ is a decomposition group of z' in $\pi_1^{\text{temp}}(X_{i,N})$. According to proposition 4.2, there exists $\alpha \in (\mathbf{Z}/p^n\mathbf{Z})^*$ such that, if β_1 is a preimage of $Y_{1,n}$ in $\text{Hom}(\pi_1^{\text{temp}}(X_{1,N}), \mu_{p^n})$, then $\alpha\beta_1\psi_N^{-1}$ is a preimage of $Y_{2,n}$ in $\text{Hom}(\pi_1^{\text{temp}}(X_{2,N}), \mu_{p^n})$. Since $\text{Ker}(\alpha\beta_1\psi_N^{-1}) = \psi(\text{Ker}(\beta_1))$ and $D_{\bar{\psi}(z')} = \psi_N(D_{z'})$, one gets that $Y_{1,n}$ is split at z' if and only if $Y_{2,n}$ is split at $\bar{\psi}(z')$, and therefore $z_{2,n} = \bar{\psi}(z_{1,n})$.

Let $c_0 \in C(\mathbb{T}_1, \mathbf{Z})^{\Gamma}$ be defined by $c_0(e_{1,i}) = 0$ and $c_0(e'_{1,i}) = 1$ for every $i \in \mathbf{Z}$. Then $\alpha_1(c_0)(x) = x$ and $\alpha_2(g^*c_0)(x) = x$. Let $\phi_{1,n}$ (resp. $\phi_{2,n}$) be a preimage of $c_0 \bmod p^n$ (resp. $g^*c_0 \bmod p^n$) by the map $\text{Hom}(\pi_1^{\text{temp}}(X_1, \tilde{X}_1), \mu_{p^n}) = H^1(X_1, \mu_{p^n}) \rightarrow C(\mathbb{T}_1, \mu_{p^n})^{\Gamma}$ (resp. $\text{Hom}(\pi_1^{\text{temp}}(X_2, \tilde{X}_2), \mu_{p^n}) = H^1(X_2, \mu_{p^n}) \rightarrow C(\mathbb{T}_2, \mu_{p^n})^{\Gamma}$). Let $z'_{1,n} = b_{z, |z|_p^{-n-\frac{1}{p-1}}} \in \mathbf{G}_m$ and $z'_{2,n} = b_{\bar{\psi}(z), |\bar{\psi}(z)|_p^{-n-\frac{1}{p-1}}} \in \mathbf{G}_m$. According to [9, lem. 4.2], $z'_{1,n}$ is characterized in \mathbf{G}_m by the fact that c_{can, p^n} is not split at $z'_{1,n}$ but is split above $[z, z'_{1,n})$. Therefore $z'_{1,n}$ is also characterized by the fact that $D_{z'_{1,n}} \not\subset \text{Ker} \phi_{1,n}$ and $D_{z'} \subset \text{Ker} \phi_{1,n}$ for every $z' \in [z, z'_{1,n})$. Similarly, $z'_{2,n}$ is characterized by the fact that $D_{z'_{2,n}} \not\subset \text{Ker} \phi_{2,n}$ and $D_{z'} \subset \text{Ker} \phi_{2,n}$ for every $z' \in [z, z'_{2,n})$. According to 4.2, one can choose $\phi_{2,n}$ to be $\alpha\phi_{1,n}\psi^{-1}$, so that $\text{Ker} \phi_{2,n} = \psi(\text{Ker} \phi_{1,n})$. Therefore, since $\bar{\psi}$ is compatible with decomposition groups, $z'_{2,n} = \bar{\psi}(z'_{1,n})$.

Using proposition 2.3, one gets

$$\begin{aligned} \text{ord}_z(\delta_1(c)) + 1 &= \lim_n \frac{1}{n} \inf \{m, z'_{1,m} \in [z, z_{1,n}]\} \\ &= \lim_n \frac{1}{n} \inf \{m, \bar{\psi}(z'_{1,m}) \in [\bar{\psi}(z), \bar{\psi}(z_{1,n})]\} \\ &= \lim_n \frac{1}{n} \inf \{m, z'_{2,m} \in [\bar{\psi}(z), z_{2,n}]\} \\ &= \text{ord}_{\bar{\psi}(z)}(\delta_2(g^*c)) + 1. \end{aligned}$$

□

Let $\mu : \mathbf{N}_{>0} \rightarrow \{-1, 0, 1\}$ be the Moebius function. Let $n \geq 1$. Let $c_n \in C(\mathbb{T}_1, \mathbf{Z})$ defined by

- $c_n(e_{1,j}) = 0$ if $j \leq 0$;
- $c_n(e'_{1,j}) = 0$ if $j \leq 0$;
- $c_n(e_{1,j}) = \mu(\frac{j}{n})$ if $j \geq 1$ and $j = 0 \bmod n$;
- $c_n(e_{1,j}) = 0$ if $j \geq 1$ and $j \neq 0 \bmod n$;
- $c_n(e'_{1,j}) = \sum_{k=1}^{\lfloor \frac{j}{n} \rfloor} \mu(k)$ if $j \geq 1$.

The associated differentials are:

$$\begin{aligned}\delta_1(c_n) &= \sum_{j \geq 1} \mu(j) \left(\frac{1}{x - q_1^{nj}} - 1 \right) dx \\ \delta_2(g^*(c_n)) &= \sum_{j \geq 1} \mu(j) \left(\frac{1}{x - q_2^{nj}} - 1 \right) dx.\end{aligned}$$

By evaluating $\delta_1(c_n)$ and $\delta_2(g^*(c_n))$ at 1 in $\mathbf{C}_p dx$, one gets

$$\delta_1(c_n)(1) = \sum_{j \geq 1} \mu(j) \frac{q_1^{jn}}{1 - q_1^{jn}} dx = \sum_{j \geq 1} \sum_{k \geq 1} \mu(j) q_1^{k j n} dx = \sum_{d \geq 1} \sum_{j | d} \mu(j) q_1^{d n} dx = q_1^n dx,$$

and similarly $\delta_2(g^*(c_n))(1) = q_2^n dx$. Let c_0 be defined by $c_0(e_{1,j}) = 0$ and $c_0(e_{1,j}) = 1$. Then $\delta_1(c_0) = \delta_2(g^*(c_0)) = \frac{dx}{x}$ and $\delta_1(c_0)(1) = \delta_2(g^*(c_0))(1) = dx$.

If $P = \sum_{n \geq 0} a_n X^n \in \mathbf{Z}_p[X]$, let $c_P = \sum_{n \neq 0} a_n c_n \in C(\mathbb{T}_1, \mathbf{Z}_p)$, so that $\delta_1(c_P)(1) = P(q_1)dx$ and $\delta_2(g^*(c_P))(1) = P(q_2)dx$.

According to lemma 4.3, $\delta_1(c_P)(1) = 0$ if and only if $\delta_2(g^*(c_P))(1) = 0$. Therefore $P(q_1) = 0$ if and only if $P(q_2) = 0$ for every $P \in \mathbf{Z}_p[X]$, which implies the result. \square

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